CYCLIC POLYNOMIALS IN DIRICHLET-TYPE SPACES OF THE UNIT BIDISK

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ABSTRACT. For $\alpha \in \mathbb{R}$, we consider the scale of function spaces, namely the Dirichlet-type space $\mathcal{D}_\alpha$ consisting of holomorphic functions on the unit bidisk $\mathbb{D}^2$, $f(z,w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

In this paper, we solve an open problem posed by Torkinejad Ziarati concerning the cyclicity of the polynomial $2-z_1-z_2$ in $\mathcal{D}_\alpha$ for $\frac{3}{2} < \alpha \le 2$. We provide an affirmative answer and, as a consequence, complete the characterization of cyclic polynomials in $\mathcal{D}_\alpha$.

1. INTRODUCTION

Let $\mathbb{C}$ denote the complex plane, $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ the open unit disk and $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ the unit circle in the complex plane. Given a polynomial $p \in \mathbb{C}[z_1, z_2]$, its bidegree is the pair (m, n), where m is the highest degree of p in the variable z_1, and n is the highest degree of p in the variable z_2. We write

$$\mathcal{Z}(p) := \{(z_1, z_2) \in \mathbb{C}^2 : p(z_1, z_2) = 0\}$$

for the vanishing set of p. A nonzero polynomial p is said to be *irreducible* if $p = qr$ with $q, r \in \mathbb{C}[z_1, z_2]$ implies that $q \in \mathbb{C}$ or $r \in \mathbb{C}$.

In the classical Hardy space $H^2(\mathbb{D})$, a function f is called *cyclic* if the smallest closed, shift-invariant subspace generated by its polynomial multiples coincide with the whole space. By Beurling's theorem, cyclic functions in this setting are precisely the outer functions, i.e., $f(0) \neq 0$ and

$$\log |f(0)| = \int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi},$$

making the theory both elegant and complete. In contrast, for the Hardy space on the bidisk $H^2(\mathbb{D}^2)$, cyclic functions are outer but there exists an outer function which is not cyclic [18]. With the present understanding, a characterization of cyclic functions seems to be a harder problem in several variables. However, cyclic polynomials are characterized in $H^2(\mathbb{D}^n)$; they

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are precisely those polynomials which do not have zeros on the polydisk $\mathbb{D}^n$ [17]. This gap between the univariate and multivariate settings motivates investigations of cyclicity in other spaces, such as the Dirichlet space and the Dirichlet-type spaces.

In the Dirichlet space D of the unit disk, Brown and Shields conjectured [6, Question 12] that a function $f \in D$ is cyclic if and only if it is outer and its boundary zero set has logarithmic capacity zero. They were able to establish the forward direction, while the converse, despite several attempts, remains open till now. Several partial results are known [9, 7]. The Brown–Shields conjecture continues to be a central open problem, motivating further exploration of cyclicity in Dirichlet-type spaces [1, 2, 3, 15].

1.1. **Dirichlet-type spaces.** We now introduce the following Dirichlet-type space of the unit bidisk, where we will investigate the cyclicity of polynomials. For $\alpha \in \mathbb{R}$, the Dirichlet-type space denoted by $\mathcal{D}_\alpha$ on $\mathbb{D}^2$ consists of all holomorphic functions $f(z,w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\|f\|_\alpha^2 := \sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

Note that for $\alpha = 0$, we recover the Hardy space $H^2(\mathbb{D}^2)$ of the unit bidisk and for $\alpha = 1$, the space $\mathcal{D}_1$ was introduced in [4] in connection with toral 2-isometries. This space has also been considered by Torkinejad Ziarati in [19], where the weight $(k+l+1)^\alpha$ is replaced by $(k+l+2)^\alpha$. However, it can be seen that these two norms are equivalent. Indeed, for $\alpha \geq 0$ we have

$$(k+l+1)^\alpha \leq (k+l+2)^\alpha \leq 2^\alpha (k+l+1)^\alpha,$$

while for $\alpha \leq 0$ it holds that

$$2^\alpha (k+l+1)^\alpha \leq (k+l+2)^\alpha \leq (k+l+1)^\alpha.$$

Investigations in this paper are motivated by the cyclicity results [2, 3] obtained for the Dirichlet-type space $\mathfrak{D}_\alpha$, $\alpha \in \mathbb{R}$, on $\mathbb{D}^2$, which consists of holomorphic functions $f(z,w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\|f\|_{\mathfrak{D}_\alpha}^2 := \sum_{k,l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{kl}|^2 < \infty. \tag{1.1}$$

From the norm definitions, it is straightforward to see that for $\alpha \geq 0$ we have

$$\|\cdot\|_\alpha \leq \|\cdot\|_{\mathfrak{D}_\alpha} \leq \|\cdot\|_{2\alpha} \quad (\mathcal{D}_{2\alpha} \subseteq \mathfrak{D}_\alpha \subseteq \mathcal{D}_\alpha), \tag{1.2}$$

and for $\alpha \leq 0$,

$$\|\cdot\|_{2\alpha} \leq \|\cdot\|_{\mathfrak{D}_\alpha} \leq \|\cdot\|_\alpha \quad (\mathcal{D}_\alpha \subseteq \mathfrak{D}_\alpha \subseteq \mathcal{D}_{2\alpha}). \tag{1.3}$$

Inequalities (1.2), (1.3) allow us to transfer properties between both kinds of Dirichlet-type spaces. In particular, we focus here on the notion of cyclicity.

Note that both spaces $\mathcal{D}_\alpha$ and $\mathfrak{D}_\alpha$ can be viewed as a generalization of the univariate Dirichlet space $D_\alpha, \alpha \in \mathbb{R}$, which consists of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, such that

$$\|f\|_{D_\alpha}^2 := \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty. \tag{1.4}$$

1.2. **Cyclic functions.** In this subsection, we explain the motivation for studying cyclic functions in the Dirichlet-type space $\mathcal{D}_\alpha$. Let (M_{z_1}, M_{z_2}) be a pair of shift operators acting on $\mathcal{D}_\alpha$, i.e.,

$$(M_{z_1}f)(z_1,z_2) := z_1 f(z_1,z_2), \quad (M_{z_2}f)(z_1,z_2) := z_2 f(z_1,z_2), \quad f \in \mathcal{D}_\alpha.$$

It is straightforward to verify that both M_{z_1} and M_{z_2} are bounded linear operators on $\mathcal{D}_\alpha$. From the operator-theoretic point of view, an important problem is to describe the closed subspaces of $\mathcal{D}_\alpha$ that are invariant under these shifts, namely those $\mathcal{M} \subseteq \mathcal{D}_\alpha$ for which

$$M_{z_1}\mathcal{M} \subseteq \mathcal{M} \qquad \text{and} \qquad M_{z_2}\mathcal{M} \subseteq \mathcal{M}.$$

A key step towards this description is to understand when a function $f \in \mathcal{D}_\alpha$ is cyclic, i.e., when the closed linear span

$$[f] := \overline{\operatorname{span}}\{z_1^k z_2^\ell f : k, \ell \geq 0\}$$

coincides with the entire space $\mathcal{D}_\alpha$. It is clear from the definition that $[f]$ is the smallest closed subspace that contains f and is invariant under the shift operators M_{z_1} and M_{z_2}. Clearly, at least one cyclic vector always exists, e.g., the constant function $f(z_1,z_2) \equiv 1$ is cyclic, because polynomials in two variables are dense in $\mathcal{D}_\alpha$. In the next section, we will see that a necessary condition for cyclicity is that f has no zeros in $\mathbb{D}^2$.

Note that if $g \in [f]$, then $[g] \subseteq [f]$. Thus to check f is cyclic in $\mathcal{D}_\alpha$, it suffices to show that there exists a sequence of polynomials $p_n \in \mathbb{C}[z_1,z_2]$ such that

$$\|p_n f - 1\|_\alpha \to 0 \quad \text{as } n \to \infty.$$

1.3. **Cyclic polynomials in $\mathfrak{D}_\alpha$.** Recent work of Bénéteau et al. [3, Theorem] provides a complete characterization of cyclic polynomials in $\mathfrak{D}_\alpha$, $\alpha \in \mathbb{R}$, on the bidisk (see [15] for the anisotropic setting $\mathfrak{D}_{(\alpha_1,\alpha_2)}$ obtained by replacing the weights $(k+1)^\alpha(l+1)^\alpha$ in (1.1) with the weights $(k+1)^{\alpha_1}(l+1)^{\alpha_2}$ and [12] for the unit ball in $\mathbb{C}^2$). Their main result shows that the cyclicity of an irreducible polynomial depends intricately on the structure of its zero set on the distinguished boundary $\mathbb{T}^2$. In particular, while non-vanishing in the bidisk is necessary for cyclicity, additional restrictions on the boundary zero set become decisive when the parameter α of the Dirichlet-type space $\mathfrak{D}_\alpha$ lies in the range $(\frac{1}{2}, \infty)$. We recall the result for the reader's convenience.

Theorem 1.1 ([3, Theorem]). *Let $p \in \mathbb{C}[z_1,z_2]$ be an irreducible polynomial with no zeros in the bidisk. We have the following:*

- *(i) If $\alpha \leq \frac{1}{2}$, then p is cyclic in $\mathfrak{D}_\alpha$.*
- *(ii) If $\frac{1}{2} < \alpha \leq 1$, then p is cyclic in $\mathfrak{D}_\alpha$ if and only if $\mathbb{Z}(p) \cap \mathbb{T}^2$ is empty or finite or p is a constant multiple of $\zeta - z_1$ or of $\zeta - z_2$ for some $\zeta \in \mathbb{T}$.*
- *(iii) If $\alpha > 1$, then p is cyclic in $\mathfrak{D}_\alpha$ if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty.*

1.4. **Cyclic polynomials in $\mathcal{D}_\alpha$.** The question of characterization of cyclic polynomials in $\mathcal{D}_\alpha$ was very recently studied by Torkinejad Ziarati [19], leaving one case open which depends on the following question posed by the author (see [19, Open problem 1]).

Question 1.2. For $\frac{3}{2} \leq \alpha \leq 2$, determine whether the polynomial $2 - z_1 - z_2$ is cyclic in $\mathcal{D}_\alpha$.

In this paper, we answer Question 1.2, which, together with [19, Theorem 31], completes the characterization of cyclic polynomials in $\mathcal{D}_\alpha$. We also provide alternative proofs for several results from [19, Section 5].

Theorem 1.3. *Let $p \in \mathbb{C}[z_1, z_2]$ be an irreducible polynomial with no zeros in the bidisk. We have the following:*

(i) If $\alpha \leq 1$, then p is cyclic in $\mathcal{D}_\alpha$.
(ii) If $1 < \alpha \leq 2$, then p is cyclic in $\mathcal{D}_\alpha$ if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty or finite.
(iii) If $\alpha > 2$, then p is cyclic in $\mathcal{D}_\alpha$ if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty.

Remark 1.4. Theorem 1.3(*iii*) follows directly from Theorem 1.1(*iii*) together with (1.2). In contrast, establishing parts (*i*) and (*ii*) of Theorem 1.3 is less straightforward. For the proof of part (*i*), we refer to [19]. The proof of part (*ii*) relies on the techniques used in the proof of [3, Theorem 3.1], together with arguments from [15, Appendix A].

1.5. **Cyclic polynomials in $\mathfrak{D}_\alpha$ versus $\mathcal{D}_\alpha$.** The following table summarizes the conditions for an irreducible polynomial p to be cyclic in the spaces $\mathfrak{D}_\alpha$ and $\mathcal{D}_\alpha$.

TABLE 1. Assume that $p \in \mathbb{C}[z_1, z_2]$ is an irreducible polynomial with no zeroes on $\mathbb{D}^2$. The table gives conditions on the set $\mathcal{Z}(p) \cap \mathbb{T}^2$ for p to be cyclic in the product-weight space $\mathfrak{D}_\alpha$ and the sum-weight space $\mathcal{D}_\alpha$ for different choices of α.

α \ space	$\mathfrak{D}_\alpha$ (weights $(k+1)^\alpha(l+1)^\alpha$)	$\mathcal{D}_\alpha$ (weights $(k+l+1)^\alpha$)
$\alpha \leq \frac{1}{2}$	no condition	no condition
$\frac{1}{2} < \alpha \leq 1$	$\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty or finite, or p is a constant multiple of $\zeta - z_1$ or $\zeta - z_2$ for some $\zeta \in \mathbb{T}$	no condition
$1 < \alpha \leq 2$	$\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$	$\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty or finite
$\alpha > 2$	$\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$	$\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$

Remark 1.5. The table highlights a clear difference between the cyclicity criteria in the two families of spaces. For the product-weight spaces $\mathfrak{D}_\alpha$, the transition occurs already at $\alpha = \frac{1}{2}$: when $\alpha \leq \frac{1}{2}$, every irreducible polynomial with no zeros in $\mathbb{D}^2$ is cyclic, whereas for $\frac{1}{2} < \alpha \leq 1$ one must already impose restrictions on the set $\mathcal{Z}(p) \cap \mathbb{T}^2$. In that range, however, the spaces $\mathfrak{D}_\alpha$ still do not distinguish between a polynomial with finitely many zeros on $\mathbb{T}^2$ and the special factors $1 - z_1$ and $1 - z_2$, which have infinitely many zeros on $\mathbb{T}^2$. In fact, the cyclicity of these latter examples was used in [3] to deduce cyclicity for polynomials with finitely many boundary zeros.

By contrast, for the sum-weight spaces $\mathcal{D}_\alpha$, no boundary condition is needed for the entire range $\alpha \leq 1$. Thus, compared with $\mathfrak{D}_\alpha$, cyclicity persists in $\mathcal{D}_\alpha$ up to the larger threshold $\alpha = 1$. A second difference appears for $1 < \alpha \leq 2$: in $\mathcal{D}_\alpha$, cyclicity still holds whenever $\mathcal{Z}(p) \cap \mathbb{T}^2$ is finite, while in $\mathfrak{D}_\alpha$ one already requires $\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$. Finally, for $\alpha > 2$ the two scales of spaces exhibit the same behavior, namely that cyclicity is equivalent to the absence of zeros on $\mathbb{T}^2$.

In particular, the table shows that the spaces $\mathcal{D}_\alpha$ are, in this sense, more permissive than the spaces $\mathfrak{D}_\alpha$: finite intersections with $\mathbb{T}^2$ remain admissible for a larger range of parameters, and the exceptional role of the factors $1-z_1$ and $1-z_2$ disappears.

1.6. **Organization of the paper.** In Section 2, we recall some definitions and necessary results needed to prove our main result. In Section 3, we present a solution to Question 1.2. In Section 4, we provide a proof of Theorem 1.3. This is done by first proving the case of cyclicity of the polynomial having at most finitely many zeros on $\mathbb{T}^2$ in $\mathcal{D}_\alpha, \alpha \leq 2$. We then complete the proof of Theorem 1.3 using the results of the preceding sections. In Section 5, we present a few necessary conditions for the cyclicity of a function in $\mathcal{D}_\alpha, \alpha \in \mathbb{R}$. In Section 6, we conclude the paper with a discussion on the cyclicity of $z_i - a$, with $|a| \geq 1$, $1 - z_1 z_2$ and a result based on capacity.

2. Preliminaries

In this section, we list some properties of cyclic functions which are needed to give a self-contained treatment of the proof of Theorem 1.3 (see Subsection 2.1) and recall an inequality on the value of a real analytic function (see Subsection 2.2).

2.1. **Some properties of cyclic functions in $\mathcal{D}_\alpha$.** Note that $\mathcal{D}_\alpha, \alpha \in \mathbb{R}$, is a reproducing kernel Hilbert space, i.e., the evaluation map e_w, for $w \in \mathbb{D}^2$, $e_w(f) := f(w)$ is a continuous linear functional on $\mathcal{D}_\alpha$ and its multiplier space is defined as

$$M(\mathcal{D}_\alpha) = \{\phi : \mathbb{D}^2 \to \mathbb{C} : \phi f \in \mathcal{D}_\alpha \text{ for all } f \in \mathcal{D}_\alpha\}.$$

Elements of $M(\mathcal{D}_\alpha)$ are called *multipliers* of $\mathcal{D}_\alpha$. It is easy to verify that all polynomials are multipliers of $\mathcal{D}_\alpha, \alpha \in \mathbb{R}$.

Some properties of cyclic functions are the following:

(i) A cyclic function in $\mathcal{D}_\alpha$ cannot vanish on the bidisk. To see this, take p_n to be a sequence of polynomials such that $\|p_n f - 1\|_\alpha \to 0$. Since evaluations are continuous, the conclusion follows from the following expression

$$p_n(z_1, z_2) f(z_1, z_2) - 1 = e_{(z_1,z_2)}(p_n f - 1).$$

(ii) Given a cyclic function $f \in \mathcal{D}_\alpha, \alpha \in \mathbb{R}$, the function defined by $g(z_1, z_2) := f(\zeta z_1, \eta z_2)$, where $\zeta, \eta \in \mathbb{T}$, is clearly also cyclic in $\mathcal{D}_\alpha$. Indeed, if there exists a sequence of polynomials $\{p_n\}_{n \in \mathbb{Z}_+}$ such that

$$\|p_n f - 1\|_\alpha \to 0, \tag{2.1}$$

then the sequence defined by $q_n(z_1, z_2) := p_n(\zeta z_1, \eta z_2)$ satisfies

$$\|q_n g - 1\|_\alpha = \|p_n f - 1\|_\alpha \to 0,$$

proving the cyclicity of g.

(iii) Assume that f is a reducible polynomial with $f = gh$ for some nonconstant polynomials g and h. Then $f = gh$ is cyclic in $\mathcal{D}_\alpha$ if and only if g and h are cyclic in $\mathcal{D}_\alpha$. Let us verify:

If f is cyclic, then there is a sequence of polynomials $\{p_n\}_{n\in\mathbb{Z}_+}$ satisfying (2.1). But then the sequence $\{r_n\}_{n\in\mathbb{Z}_+}$ and $\{s_n\}_{n\in\mathbb{Z}_+}$, where $r_n := p_n g$ and $s_n := p_n h$, satisfy $\|r_n h - 1\|_\alpha \to 0$ and $\|s_n g - 1\|_\alpha \to 0$, proving cyclicity of g and h.

Conversely, assume that g and h are cyclic. Then there exists a sequence of polynomials $\{r_n\}_{n\in\mathbb{Z}_+}$ such that $\|r_n g - 1\|_\alpha \to 0$. Note that

$$\|r_n gh - h\|_\alpha \leq \|M_h\| \|r_n g - 1\|_\alpha$$

where $\|M_h\|$ is the operator norm of the multiplication operator M_h. This shows that $h \in [f]$. Hence $[h] \subseteq [f]$. Since h is cyclic, it follows that f is cyclic.

Therefore, it suffices to characterize cyclicity of irreducible polynomials in $\mathcal{D}_\alpha$.

2.2. **Łojasiewicz's inequality.** In the proof of Theorem 4.1 below, the following inequality will be used essentially.

Theorem 2.1 ([16, Łojasiewicz's inequality]). *Let f be a nonzero real analytic function on an open set $U \subseteq \mathbb{R}^n$. Assume the zero set $\mathcal{Z}(f)$ of f in U is nonempty. Let E be a compact subset of U. Then there are constants $C > 0$ and $q \in \mathbb{N}$, depending on E, such that*

$$|f(x)| \geq C \cdot \operatorname{dist}(x, \mathcal{Z}(f))^q$$

for every $x \in E$.

3. Cyclicity of the polynomial $2 - z_1 - z_2$

In this section, we answer Question 1.2 using the techniques from [15, Appendix A].

Theorem 3.1. *Let $\alpha \leq 2$. Then $p(z_1, z_2) = 2 - z_1 - z_2$ is cyclic in $\mathcal{D}_\alpha$.*

Proof. It suffices to show that p is cyclic in $\mathcal{D}_2$. Let $f \in \mathcal{D}_2$ be such that $f \perp [p]$. Consider the following series of f,

$$f(z_1, z_2) = \sum_{i,j=0}^{\infty} \frac{b_{i,j}}{(i+j+1)^2} z_1^i z_2^j.$$

We will show that $f = 0$, or equivalently $b_{i,j} = 0$ for $i, j \in \mathbb{Z}_+$. By $f \perp [p]$, it follows that

$$2b_{k,l} = b_{k+1,l} + b_{k,l+1}, \quad k, l \in \mathbb{Z}_+. \tag{3.1}$$

Since $f \in \mathcal{D}_2$, a new function

$$g(z_1, z_2) := \sum_{i,j\geq 0} b_{i,j} z_1^i z_2^j$$

belongs to $\mathcal{D}_{-2}$, and by (3.1),

$$(z_1 + z_2 - 2z_1 z_2) g(z_1, z_2) = z_1 g(z_1, 0) + z_2 g(0, z_2), \quad (z_1, z_2) \in \mathbb{D}^2. \tag{3.2}$$

We next introduce the substitutions

$$z_1 = \frac{\zeta}{\zeta - 1}, \qquad z_2 = \frac{\zeta}{\zeta + 1}. \tag{3.3}$$

Observe that $z_1 \in \mathbb{D}$ precisely when $\Re\zeta < \frac{1}{2}$, and $z_2 \in \mathbb{D}$ precisely when $\Re\zeta > -\frac{1}{2}$, where $\Re\zeta$ denotes the real part of the complex number ζ. By substituting the above expressions for z_1 and z_2 into (3.2), we get

$$0 = \frac{\zeta}{\zeta - 1} g\left(\frac{\zeta}{\zeta - 1}, 0\right) + \frac{\zeta}{\zeta + 1} g\left(0, \frac{\zeta}{\zeta + 1}\right), \qquad \text{for } -\tfrac{1}{2} < \Re\zeta < \tfrac{1}{2}.$$

Thus, defining $h : \mathbb{C} \to \mathbb{C}$ by

$$h(\zeta) = \begin{cases} \dfrac{1}{\zeta - 1}\, g\left(\dfrac{\zeta}{\zeta - 1}, 0\right), & \text{if } \Re\zeta < \frac{1}{2}, \\ -\dfrac{1}{\zeta + 1}\, g\left(0, \dfrac{\zeta}{\zeta + 1}\right), & \text{if } \Re\zeta > -\frac{1}{2}, \end{cases}$$

we obtain that h is a well-defined entire function. Note that

$$\sum_{k \geq 0} \frac{|b_{k0}|^2}{(k+1)^2} \asymp \int_{\mathbb{D}} |g(z_1, 0)|^2 (1 - |z_1|^2)\, dA(z_1).$$

This can be verified by integrating the right-hand side using polar coordinates. Thus, we have

$$\begin{aligned} \sum_{k \geq 0} \frac{|b_{k0}|^2}{(k+1)^2} &\asymp \int_{\Re\zeta < 1/2} |(\zeta - 1)h(\zeta)|^2 \left(1 - \left|\frac{\zeta}{\zeta - 1}\right|^2\right) \frac{dA(\zeta)}{|\zeta - 1|^4} \\ &= \int_{\Re\zeta < 1/2} |h(\zeta)|^2 \frac{1 - 2\Re\zeta}{|\zeta - 1|^4}\, dA(\zeta), \end{aligned} \tag{3.4}$$

and similarly,

$$\begin{aligned} \sum_{l \geq 0} \frac{|b_{0l}|^2}{(l+1)^2} &\asymp \int_{\mathbb{D}} |g(0, z_2)|^2 (1 - |z_2|^2)\, dA(z_2) \\ &= \int_{\Re\zeta > -1/2} |(\zeta + 1)h(\zeta)|^2 \left(1 - \left|\frac{\zeta}{\zeta + 1}\right|^2\right) \frac{dA(\zeta)}{|\zeta + 1|^4} \\ &= \int_{\Re\zeta > -1/2} |h(\zeta)|^2 \frac{1 + 2\Re\zeta}{|\zeta + 1|^4}\, dA(\zeta). \end{aligned}$$

Both series are finite due to $g \in \mathcal{D}_{-2}$; hence, the sum of the two integrals is finite, and consequently,

$$\int_{|\zeta| > 1} \frac{|h(\zeta)|^2}{|\zeta|^4}\, dA(\zeta) < \infty.$$

This forces h to be a polynomial of degree at most 1, i.e.,

$$h(\zeta) = a(\zeta - 1) + b, \quad a, b \in \mathbb{C}. \tag{3.5}$$

Assume that $a \neq 0$. Using (3.5) in (3.4), we get

$$\int_{\Re\zeta < 1/2} \frac{1 - 2\Re\zeta}{|\zeta - 1|^2}\, dA(\zeta) < \infty,$$

which is a contradiction. Hence, $a = 0$. Note that

$$\begin{aligned} g(z_1, 0) \underbrace{=}_{(3.3)} \frac{1}{z_1 - 1} h\left(\frac{z_1}{z_1 - 1}\right) &= \frac{b}{z_1 - 1}, \\ g(0, z_2) \underbrace{=}_{(3.3)} \frac{1}{z_2 - 1} h\left(\frac{z_2}{1 - z_2}\right) &= \frac{b}{z_2 - 1}. \end{aligned} \tag{3.6}$$

Using (3.6) in (3.2), we get

$$(z_1 + z_2 - 2z_1z_2)g(z_1, z_2) = \frac{bz_1}{z_1 - 1} + \frac{bz_2}{z_2 - 1}, \quad (z_1, z_2) \in \mathbb{D}^2. \tag{3.7}$$

We have

$$g(z_1, z_2) = -\frac{b}{(1 - z_1)(1 - z_2)}.$$

For $|z_1| < 1$ and $|z_2| < 1$, the geometric series gives

$$\frac{1}{1 - z_1} = \sum_{k=0}^{\infty} z_1^k, \qquad \frac{1}{1 - z_2} = \sum_{l=0}^{\infty} z_2^l,$$

hence, by multiplication of absolutely convergent series,

$$\frac{1}{(1 - z_1)(1 - z_2)} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} z_1^k z_2^l.$$

Therefore

$$g(z_1, z_2) = -b \sum_{k,l=0}^{\infty} z_1^k z_2^l,$$

so the Taylor coefficients of g satisfy

$$a_{kl} = -b, \qquad k, l \geq 0.$$

It follows that

$$\|g\|_{-2}^2 = \sum_{k,l=0}^{\infty} (k + l + 1)^{-2} |a_{kl}|^2 = |b|^2 \sum_{k,l=0}^{\infty} \frac{1}{(k + l + 1)^2}.$$

We now group terms by $n = k + l$:

$$\sum_{k,l=0}^{\infty} \frac{1}{(k + l + 1)^2} = \sum_{n=0}^{\infty} \sum_{\substack{k,l \geq 0 \\ k+l=n}} \frac{1}{(n + 1)^2}.$$

For each $n \geq 0$ there are exactly $n+1$ pairs $(k, l) \in \mathbb{N}_0^2$ with $k+l = n$. Hence

$$\sum_{n=0}^{\infty} \sum_{\substack{k,l \geq 0 \\ k+l=n}} \frac{1}{(n + 1)^2} = \sum_{n=0}^{\infty} \frac{n + 1}{(n + 1)^2} = \sum_{n=0}^{\infty} \frac{1}{n + 1}.$$

The latter series diverges, and therefore

$$\|g\|_{-2} = \infty.$$

Thus, $g \notin \mathcal{D}_{-2}$ unless $b = 0$. In particular, if $g \in \mathcal{D}_{-2}$, we must have $b = 0$, which forces $g = 0$. Therefore, $h = 0$, and consequently $f = 0$. This completes the proof. □

Remark 3.2. The proof shows that if a sequence of complex numbers $\{b_{k,l}\}_{k,l\in\mathbb{Z}_+}$ satisfies (3.1) and

$$\sum_{k,l\in\mathbb{Z}_+} \frac{|b_{k,l}|^2}{(k+l+1)^2} < \infty,$$

then $b_{k,l} = 0$ for all $k, l \in \mathbb{Z}_+$.

The following result is an immediate and noteworthy consequence, which will be used essentially in the proof of Theorem 4.1 below.

Corollary 3.3. *For $\zeta_1, \zeta_2 \in \mathbb{T}$, $p(z_1, z_2) = 2 - \zeta_1 z_1 - \zeta_2 z_2$ is cyclic in $\mathcal{D}_\alpha$ for $\alpha \le 2$.*

Proof. This follows from Theorem 3.1 together with the fact that cyclicity is preserved under rotation. □

4. A proof of Theorem 1.3

First, we investigate the cyclicity of polynomials in $\mathcal{D}_\alpha$, $\alpha \le 2$, that possess only finitely many zeros on $\mathbb{T}^2$. In the previously studied settings (see [3, Section 3]), the cyclicity of the polynomial $1 - z_i$ was essentially used in the proof of the main result [3, Theorem 3.1]. However, in our framework, $1 - z_i$ needs to be replaced by another polynomial cyclic in $\mathcal{D}_\alpha$, $\alpha \le 2$, since $1 - z_i$ is cyclic in $\mathcal{D}_\alpha$ only for $\alpha \le 1$ (see [19, Proposition 38] or Proposition 6.1 below). It turns out that $2 - z_1 - z_2$ serves as a suitable substitute.

Theorem 4.1. *Consider a polynomial $p \in \mathbb{C}[z_1, z_2]$ having no zeros in $\mathbb{D}^2$ and finitely many on $\mathbb{T}^2$. Then p is cyclic in $\mathcal{D}_\alpha$ for $\alpha \le 2$.*

The proof of Theorem 4.1 will parallel the arguments in the proof of [3, Theorem 3.2], which deals with the cyclicity in $\mathfrak{D}_\alpha$, using the modifications described in the paragraph before Theorem 4.1.

The following result, which is an analog of [3, Lemma 3.3], will allow us to compare polynomials having finitely many zeros on $\mathbb{T}^2$ with the polynomials of the type $2-\zeta z_1 - \eta z_2$, $\zeta, \eta \in \mathbb{T}$, whose cyclicity has already been established in Corollary 3.3.

Lemma 4.2. *Suppose $f \in \mathbb{C}[z_1, z_2]$ has no zeros in $\mathbb{D}^2$ and finitely many on $\mathbb{T}^2$, i.e., $\mathcal{Z}(f) \cap \mathbb{T}^2 = \{(\zeta_j, \eta_j) \in \mathbb{T}^2, j = 1, \dots, k\}$. Then, for any integer m, there exists a sufficiently large N such that the function*

$$Q(z_1, z_2) = \frac{\prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2)^N}{f(z_1, z_2)}$$

is m-times differentiable on $\mathbb{T}^2$.

Proof. Let $\{(\zeta_j, \eta_j) \in \mathbb{T}^2$, $j = 1, \dots, k$ be as in the statement of the lemma and define the polynomials

$$p_j(z_1, z_2) = 2 - {\zeta_j}^{-1} z_1 - {\eta_j}^{-1} z_2.$$

Clearly, p_j has only one zero (ζ_j, η_j) on $\mathbb{T}^2$. Note that

$$|p_j(z_1, z_2)|^2 \le |z_1 - \zeta_j|^2 + |z_2 - \eta_j|^2 + 2|(z_1 - \zeta_j)(z_2 - \eta_j)|. \tag{4.1}$$

Writing $z_1, z_2, \zeta_j, \eta_j \in \mathbb{T}$ as

$$z_1 = e^{ix_1},\ z_2 = e^{ix_2},\ \zeta_j = e^{iy_{1,j}},\ \eta_j = e^{iy_{2,j}},$$

where $x_1, x_2, y_{1,j}, y_{2,j} \in [0, 2\pi)$, respectively, (4.1) becomes

$$\begin{aligned} |p_j(e^{ix_1}, e^{ix_2})|^2 \le |e^{ix_1} - e^{iy_{1,j}}|^2 + |e^{ix_2} - e^{iy_{2,j}}|^2 \\ + 2\left|(e^{ix_1} - e^{iy_{1,j}})(e^{ix_2} - e^{iy_{2,j}})\right|. \end{aligned} \tag{4.2}$$

Define

$$r(x_1, x_2) = |f(e^{ix_1}, e^{ix_2})|^2, \qquad (x_1, x_2) \in \mathbb{R}^2,$$

and let $\mathcal{Z}(r)$ denote its zero set. Consider the compact set $E = [0, 2\pi]^2$. By Theorem 2.1, there exist a constant $C > 0$ and an integer $q \in \mathbb{N}$ such that

$$r(x) \ge C \operatorname{dist}(x, \mathcal{Z}(r))^q, \qquad x \in E.$$

Since the set $\mathcal{Z}(r) \cap E$ consists of finitely many points, there is a constant $c > 0$ satisfying

$$\operatorname{dist}(x, \mathcal{Z}(r))^2 \ge c \prod_{y \in \mathcal{Z}(r) \cap E} |x - y|^2, \qquad x \in E. \tag{4.3}$$

Also, we have

$$|x - y|^2 = \sum_{j=1,2} |x_j - y_j|^2 \ge \sum_{j=1,2} |e^{ix_j} - e^{iy_j}|^2 \ge 2 \prod_{j=1,2} |e^{ix_j} - e^{iy_j}|. \tag{4.4}$$

Using (4.2),(4.3) and (4.4), we have a constant $C_1 > 0$ such that

$$\operatorname{dist}(x, \mathcal{Z}(r))^2 \ge C_1 \prod_j |p_j|^2.$$

This yields that the function

$$\frac{\prod_j |p_j|^q}{|f|^2},$$

is bounded on $\mathbb{T}^2$. We now apply the standard trick, which is to increase the exponent in the numerator and assign the value zero at the zeros of f, to obtain a function that is m-times continuously differentiable on $\mathbb{T}^2$. This completes the proof of Lemma 4.2. □

Finally, we can prove Theorem 4.1

Proof of Theorem 4.1. By Lemma 4.2, we obtain a function

$$Q(z_1, z_2) = \frac{\prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2)^N}{p(z_1, z_2)} =: \frac{g(z_1, z_2)}{p(z_1, z_2)},$$

where $(\zeta_i, \eta_i) \in \mathbb{T}^2$, which is twice continuously differentiable on $\mathbb{T}^2$. Thus its Fourier coefficients $\widehat{Q}(k, l)$ satisfy

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k + 1)^2 (l + 1)^2 < \infty.$$

But since

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k + l + 1)^2 \le \sum_{k,l} |\widehat{Q}(k, l)|^2 (k + 1)^2 (l + 1)^2,$$

we obtain $Q \in \mathcal{D}_\alpha, \alpha \le 2$. Hence $g(z_1, z_2) \in p\mathcal{D}_\alpha, \alpha \le 2$. Since g is cyclic in $\mathcal{D}_2$ by Corollary 3.3 and p is a multiplier, we obtain that p is also cyclic in $\mathcal{D}_2$. □

4.1. **Proof of Theorem 1.3.** For completeness, we provide a proof of Theorem 1.3 in this section.

We recall the following result from [19, p. 17]. This establishes the cyclicity of polynomials having no zeros in the bidisk. The proof closely parallels the argument of [3, Theorem 4.1] which deals with the cyclicity in $\mathfrak{D}_\alpha$, with a few modifications needed in the framework of $\mathcal{D}_\alpha$.

Theorem 4.3. *Assume that $\alpha \le 1$. Any polynomial $f \in \mathbb{C}[z_1, z_2]$ that does not vanish in the bidisk is cyclic in $\mathcal{D}_\alpha$.*

The proof of Theorem 1.3 is now straightforward using the results above.

Proof of Theorem 1.3. Observe that for $\alpha > 0$,

$$(1+i)^{\alpha/2}(1+j)^{\alpha/2} \le (1+i+j)^\alpha \le (1+i)^\alpha(1+j)^\alpha, \quad i, j \ge 0. \tag{4.5}$$

We divide the argument into three cases according to the value of α: $\alpha > 2$, $\alpha \in (1, 2]$ and $\alpha \le 1$.

Case 1: $\alpha > 2$. First, assume that p is cyclic in $\mathcal{D}_\alpha$. By the first inequality in (4.5), it follows that p is cyclic in $\mathfrak{D}_{\alpha/2}$. Since $\alpha/2 > 1$, Theorem 1.1(*iii*) implies that $\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$.

Now assume that $\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$. Then, by Theorem 1.1(*iii*), p is cyclic in $\mathfrak{D}_\alpha$. Using the second inequality in (4.5), we conclude that p is cyclic in $\mathcal{D}_\alpha$.

Case 2: $\alpha \in (1, 2]$. The first inequality in (4.5) implies that if a polynomial p is cyclic in $\mathcal{D}_\alpha$, then it is also cyclic in $\mathfrak{D}_{\alpha/2}$. By Theorem 1.1(*ii*), the zero set $\mathcal{Z}(p) \cap \mathbb{T}^2$ is either empty or finite, or p is a constant multiple of $\zeta - z_1$ or $\zeta - z_2$ for some $\zeta \in \mathbb{T}$. However, by [19, Proposition 38], the functions $\zeta - z_i$, with $\zeta \in \mathbb{T}$, are not cyclic in $\mathcal{D}_\alpha$ (see also Proposition 6.1). This completes the proof of the forward implication in this case.

For the converse, we have the following:

Subcase 2.1: $\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$. Using the case $\alpha > 2$, we obtain that p is cyclic in $\mathcal{D}_3$. Since $\|\cdot\|_\alpha \le \|\cdot\|_3$ for $\alpha \le 2$, cyclicity in $\mathcal{D}_\alpha$ follows.

Subcase 2.2: $\mathcal{Z}(p) \cap \mathbb{T}^2$ is finite. This subcase follows from Theorem 4.1.

Case 3: $\alpha \le 1$. This case follows from Theorem 4.3. □

Remark 4.4. (i) The case $\alpha > 2$, has been solved in [19, Theorem 31(c)] by proving that $\mathcal{D}_\alpha$ is an algebra. We have provided an alternative proof.

(ii) Note that if p has no zeros in the bidisk and only finitely many zeros on $\mathbb{T}^2$, then its cyclicity in $\mathcal{D}_\alpha$ for $\alpha \le 1$ also follows from Theorem 1.1 by a direct comparison of norms, similarly to Case 1 in the proof of Theorem 1.3 above. However, this result excludes the cyclicity of a polynomial having infinitely many zeros on $\mathbb{T}^2$ as in the case of $1 - z_1 z_2$.

4.2. **An alternative approach to non-cyclicity in $\mathcal{D}_\alpha$ through its relation to the anisotropic Dirichlet-type spaces.** We conclude this section by presenting an alternative approach, suggested by the referee, for proving the non-cyclicity of a polynomial in $\mathcal{D}_\alpha$ using results on cyclicity in the anisotropic Dirichlet-type spaces. This approach yields the following necessary conditions for cyclicity:

(i) If $\alpha > 1$ and $p \in \mathbb{C}[z_1, z_2]$ is cyclic in $\mathcal{D}_\alpha$, then $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty or finite.
(ii) If $\alpha > 2$ and $p \in \mathbb{C}[z_1, z_2]$ is cyclic in $\mathcal{D}_\alpha$, then $\mathcal{Z}(p) \cap \mathbb{T}^2 = \varnothing$.

Let $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ be fixed. Recall that the anisotropic Dirichlet-type space $\mathfrak{D}_{(\alpha_1,\alpha_2)}$ consists of all holomorphic functions on the unit bidisk $\mathbb{D}^2$, $f(z,w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\|f\|^2_{(\alpha_1,\alpha_2)} := \sum_{k,l=0}^{\infty} (k+1)^{\alpha_1}(l+1)^{\alpha_2}|a_{kl}|^2 < \infty$$

(see [15, Subsection 1.1]).

Proof of (i). Let $\alpha > 1$. If p is cyclic in $\mathcal{D}_\alpha$, then there exists a sequence of polynomials (q_n) such that $q_n p \to 1$ in $\mathcal{D}_\alpha$. Since $\|\cdot\|_{(\alpha,0)} \le \|\cdot\|_\alpha$ for $\alpha > 0$, it follows that $q_n p \to 1$ in $\mathfrak{D}_{(\alpha,0)}$ and p is cyclic in $\mathfrak{D}_{(\alpha,0)}$. Thus, (i) follows by applying [15, Theorem 1], which characterizes cyclicity in the anisotropic spaces $\mathfrak{D}_{(\alpha_1,\alpha_2)}$, with $(\alpha_1, \alpha_2) := (\alpha, 0)$. In this case, $\alpha_1 + \alpha_2 = \alpha > 1$ and $\min\{\alpha_1, \alpha_2\} = 0 \le 1$.

Proof of (ii). Let $\alpha > 2$. Observe that for $\alpha > 0$,

$$(k+1)^{\alpha/2}(l+1)^{\alpha/2} \le (k+l+1)^{\alpha},$$

and hence $\mathcal{D}_\alpha \subseteq \mathfrak{D}_{(\alpha/2,\alpha/2)}$. Now let $p \in \mathbb{C}[z_1, z_2]$. As in the proof of (i), if p is cyclic in $\mathcal{D}_\alpha$, then it is also cyclic in $\mathfrak{D}_{(\alpha/2,\alpha/2)}$. Applying [15, Theorem 1] with $(\alpha_1, \alpha_2) = (\alpha/2, \alpha/2)$, yields (ii).

Remark 4.5. By [5, Proposition 4.5], for $0 < \alpha < 2$, we have

$$\mathcal{D}_\alpha = \mathfrak{D}_{(\alpha,0)} \cap \mathfrak{D}_{(0,\alpha)}.$$

5. General properties of cyclic functions

In this section, we discuss some general properties of cyclic functions.

5.1. **Slices of a function.**

In this subsection, we establish a result on the cyclicity of univariate slices of a function, which serves as a quick tool for determining whether a function is a suitable candidate for being cyclic.

Let $f = f(z_1, z_2)$ be a holomorphic function on the bidisk. By fixing one variable, say z_1, the slice

$$f_{z_1} : \mathbb{D} \to \mathbb{C}, \quad f_{z_1}(z_2) := f(z_1, z_2),$$

is a holomorphic function on the unit disk. The slice f_{z_2} is defined analogously.

Proposition 5.1. *If f is cyclic in $\mathcal{D}_\alpha$, then f_{z_1} and f_{z_2} are cyclic in D_α.*

Proof. Let $\alpha \geq 0$ and $z_1 \in \mathbb{D}$. Consider

$$\begin{aligned}
\|f_{z_1}\|^2_{D_\alpha} \underbrace{=}_{(1.4)} \sum_{j\geq 0}(j+1)^\alpha \left|\sum_{i\geq 0} a_{ij} z_1^i\right|^2 \\
&= \sum_{j\geq 0}\left|\sum_{i\geq 0}(j+1)^{\alpha/2} a_{ij} z_1^i\right|^2 \\
&\leq \sum_{j\geq 0}\left(\sum_{i\geq 0}(j+1)^\alpha |a_{ij}|^2\right)\left(\sum_{i\geq 0}|z_1|^{2i}\right) \\
&\leq \sum_{j\geq 0}\left(\sum_{i\geq 0}(i+j+1)^\alpha |a_{ij}|^2\right)\left(\sum_{i\geq 0}|z_1|^{2i}\right) \\
&\leq \frac{1}{1-|z_1|^2}\,\|f\|^2_\alpha,
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality.

Now suppose that $\alpha < 0$. For a fixed $z_1 \in \mathbb{D}$, we have $f_{z_1} \in D_\alpha$. Observe that

$$f_{z_1}(z_2) = f(z_1, z_2) = \sum_{i=0}^{\infty} f_i(z_1)\, z_2^i,$$

and hence

$$\|f_{z_1}\|^2_{D_\alpha} = \sum_{i=0}^{\infty}(i+1)^\alpha\, |f_i(z_1)|^2.$$

Since each $f_i \in D_\alpha$ and D_α is a reproducing kernel Hilbert space, we have

$$f_i(z_1) = \langle f_i, k_{z_1}\rangle,$$

where k_{z_1} denotes the reproducing kernel at z_1. Therefore,

$$\|f_{z_1}\|^2_{D_\alpha} = \sum_{i=0}^{\infty}(i+1)^\alpha\, |\langle f_i, k_{z_1}\rangle|^2.$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\|f_{z_1}\|_{D_\alpha} \leq \|k_{z_1}\|_{D_\alpha}\|f\|_{\mathfrak{D}_\alpha} \underbrace{\leq}_{(1.3)} \|k_{z_1}\|_{D_\alpha}\|f\|_{\mathcal{D}_\alpha}$$

(cf. [2, Proposition 2.1]).

Combining the norm estimates for both $\alpha \geq 0$ and $\alpha < 0$, we conclude that if f is cyclic in $\mathcal{D}_\alpha$, then the slices f_{z_1} and f_{z_2} are cyclic in D_α. □

A natural question is whether the converse of the above result holds. To examine this, consider $p(z_1, z_2) = 2 - z_1 - z_2$. Note that the slices of p are cyclic in D_α for all α, but p itself is not cyclic in $\mathcal{D}_\alpha$ for $\alpha > 2$ (by Theorem 1.3).

5.2. **Diagonal restriction of a function.** In this subsection, we establish a result on the cyclicity of the diagonal restriction of a function, which is another criterion for determining whether a function is a suitable candidate for being cyclic.

Given a holomorphic function f on $\mathbb{D}^2$, define the diagonal restriction $\mathcal{O}f$ by $(\mathcal{O}f)(z) := f(z,z)$. Then $\mathcal{O}f$ is holomorphic on $\mathbb{D}$. The following proposition gives a necessary condition for the cyclicity of a function in $\mathcal{D}_\alpha$. The proof follows closely the proof of [2, Proposition 2.2].

Proposition 5.2. *If f is cyclic in $\mathcal{D}_\alpha$, then $\mathcal{O}f$ is cyclic in $D_{\alpha-1}$.*

Proof. It suffices to prove that for $f \in \mathcal{D}_\alpha, \alpha \in \mathbb{R}$,

$$\|\mathcal{O}f\|_{D_{\alpha-1}} \;\leq\; \|f\|_\alpha. \tag{5.1}$$

Indeed, if $q_n f \to 1$ in $\mathcal{D}_\alpha$, then by (5.1) we have

$$\|\mathcal{O}(q_n)\mathcal{O}(f) - 1\|_{D_{\alpha-1}} = \|\mathcal{O}(q_n f - 1)\|_{D_{\alpha-1}} \leq \|q_n f - 1\|_\alpha.$$

Thus, $\mathcal{O}(f)$ is cyclic in $D_{\alpha-1}$. It remains to prove (5.1).

Let $f(z_1,z_2) = \sum_{k=0}^\infty \sum_{l=0}^\infty a_{k,l} z_1^k z_2^l$. Then

$$\mathcal{O}f(z) = \sum_{k=0}^\infty \sum_{l=0}^\infty a_{k,l} z^{k+l}$$

converges absolutely for every $|z| < 1$, hence $\mathcal{O}f$ can be rewritten as

$$\mathcal{O}f(z) = \sum_{n=0}^\infty b_n z^n, \qquad b_n = \sum_{k+l=n} a_{k,l} = \sum_{k=0}^n a_{k,n-k}.$$

Thus,

$$\|\mathcal{O}f\|^2_{D_{\alpha-1}} = \sum_{n=0}^\infty |b_n|^2 (n+1)^{\alpha-1} = \sum_{n=0}^\infty \left| \sum_{k=0}^n a_{k,n-k} \right|^2 (n+1)^{\alpha-1}. \tag{5.2}$$

By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \left| \sum_{k=0}^n a_{k,n-k} \right|^2 &\leq \left(\sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha \right) \sum_{k=0}^n (n+1)^{-\alpha} \\ &= \sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha (n+1)^{-\alpha+1}. \end{aligned}$$

Using this in (5.2), it follows that

$$\|\mathcal{O}f\|^2_{D_{\alpha-1}} \leq \sum_{n=0}^\infty \sum_{k=0}^n |a_{k,n-k}|^2 (n+1)^\alpha = \|f\|^2_\alpha,$$

proving (5.1). □

Remark 5.3. We refer the reader to ([2, Proposition 2.2]) for analogous statement for $\mathfrak{D}_\alpha$.

6. Concluding remarks

We provided a self-contained proof of the cyclicity of the polynomial $2 - z_1 - z_2$; we now give simple proofs of the cyclicity of $1 - z_1 z_2$ and $z_1 - a, |a| \geq 1$ using standard techniques. These are model polynomials and often play a crucial role in establishing cyclicity results for broader classes of functions (e.g., see the proof of [15, Theorem 5]), and therefore merit particular attention to ensure independent proofs.

The following has been proved in [19, Proposition 38] using a dilation argument. We provide an alternative proof.

Proposition 6.1. *Let $p(z) = z - a$ with $|a| \geq 1$. Then $P(z_1, z_2) := p(z_1)$ is cyclic in $\mathcal{D}_\alpha$ if and only if $\alpha \leq 1$.*

Proof. For $f \in D_\alpha$, define $F(z_1, z_2) = f(z_1)$. Note that $\|f\|_{D_\alpha} = \|F\|_{\mathcal{D}_\alpha}$. Since it is well-known [6] that p is cyclic in D_α if and only if $\alpha \leq 1$, the implication ($\Leftarrow$) is clear. For the implication ($\Rightarrow$), assume that $\alpha > 1$. Let $q_n \in \mathbb{C}[z_1, z_2]$ be a sequence such that $\|q_n F - 1\|_\alpha \to 0$. By the orthogonality of monomials in $\mathcal{D}_\alpha$, one can choose $q_n(z_1, z_2) =: g_n(z_1)$ where $g_n \in \mathbb{C}[z]$. But since $\|q_n F - 1\|_\alpha = \|g_n p - 1\|_{D_\alpha}$, this contradicts the one variable Brown-Shields theorem for $\alpha > 1$. □

Proposition 6.2. $1 - z_1 z_2$ *is cyclic in $\mathcal{D}_\alpha$ if and only if $\alpha \leq 1$.*

Proof. Note that if f is a function of one variable and F is defined by $F(z_1, z_2) := f(z_1 z_2)$, then the following inequalities hold:

$$\|f\|_{D_\alpha} \leq \|F\|_\alpha \leq 2^\alpha \|f\|_{D_\alpha}, \quad \text{for } \alpha \geq 0, \tag{6.1}$$

$$2^\alpha \|f\|_{D_\alpha} \leq \|F\|_\alpha \leq \|f\|_{D_\alpha}, \quad \text{for } \alpha < 0. \tag{6.2}$$

Let $\alpha \in \mathbb{R}$. Define

$$P(z_1, z_2) = 1 - z_1 z_2 \quad \text{and} \quad p(z) = 1 - z.$$

Assume first that $\alpha \geq 0$ and that P is cyclic in $\mathcal{D}_\alpha$. Let $\{P_n\}_n \subset \mathbb{C}[z_1, z_2]$ be a sequence of polynomials such that $\|P_n P - 1\|_\alpha \to 0$. Using the orthogonality of monomials in $\mathcal{D}_\alpha$, we may assume without loss of generality that $P_n(z_1, z_2) = p_n(z_1 z_2)$, for some sequence of one-variable polynomials $\{p_n\}_n \subset \mathbb{C}[z]$. By the first inequality in (6.1), it follows that $\|p_n p - 1\|_{D_\alpha} \to 0$, and hence $1 - z$ is cyclic in D_α. By the Brown–Shields theorem, this implies that $\alpha \leq 1$.

The converse direction follows by a similar argument, using the second inequality in (6.1).

For $\alpha < 0$, the reasoning is analogous, with (6.2) used in place of (6.1). □

We conclude the paper with a few remarks on capacity. Finite logarithmic capacity and Riesz α-capacity play important roles in identifying non-cyclic functions in $\mathfrak{D}_\alpha$. This approach originates in the work of Brown and Shields [6] and was later extended to several variables by various authors. We refer the reader to [3, Definition 2.2] for the definition of Riesz α-capacity and to [2, Definition 4.1] for the definition of finite logarithmic capacity.

Note that if $f \in \mathcal{D}_2$, then $f \in \mathfrak{D}_1$. For functions $f \in \mathfrak{D}_1$, one can apply the notion of finite logarithmic capacity to determine whether f is non-cyclic in $\mathfrak{D}_1$ [2, Proposition 4.2], and consequently not cyclic in $\mathcal{D}_2$. A similar conclusion holds for $f \in \mathcal{D}_\alpha$ with $0 < \alpha < 2$.

This discussion is summarized in the following result.

Proposition 6.3. *Let $\alpha \in (0, 2]$ and $f \in \mathcal{D}_\alpha$. Let f^* denote the radial limit of f, i.e., $f^*(e^{i\theta_1}, e^{i\theta_2}) = \lim_{r\to 1^-} f(re^{i\theta_1}, re^{i\theta_2})$ wherever the radial limit exists. Then the following holds:*

(a) If $\alpha = 2$ and $\mathcal{Z}(f^*)$ has positive logarithmic capacity, then f is not cyclic in $\mathcal{D}_2$.
(b) If $0 < \alpha < 2$ and $\mathcal{Z}(f^*)$ has positive Riesz α-capacity, then f is not cyclic in $\mathcal{D}_\alpha$.

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