CYCLIC POLYNOMIALS IN DIRICHLET-TYPE SPACES OF THE UNIT BIDISK

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Abstract. For $\alpha \in \mathbb{R}$, we consider the scale of function spaces, namely the Dirichlet-type space $\mathcal{D}_\alpha$ consisting of holomorphic functions on the unit bidisk $\mathbb{D}^2$, $f(z,w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

We present a complete characterization of cyclic polynomials in $\mathcal{D}_\alpha$, i.e., given an irreducible polynomial p, the following holds:

(i) If $\alpha \leq 1$, then p is cyclic in $\mathcal{D}_\alpha$.
(ii) If $1 < \alpha \leq 2$, then p is cyclic in $\mathcal{D}_\alpha$ if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty or finite.
(iii) If $\alpha > 2$, then p is cyclic in $\mathcal{D}_\alpha$ if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty.

1. Introduction

Let $\mathbb{C}$ denote the complex plane, $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ the open unit disk and $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ the unit circle in the complex plane. For $k \in \mathbb{N}$, we denote by $\mathbb{C}^k[z_1, z_2]$ the space of polynomials in two variables with coefficients in the column space $\mathbb{C}^k$. Given $p \in \mathbb{C}[z_1, z_2]$, its bidegree is the pair (m, n), where m is the highest degree of p in the variable z_1, and n is the highest degree of p in the variable z_2. We write

$$\mathcal{Z}(p) := \{(z_1, z_2) \in \mathbb{C}^2 \colon p(z_1, z_2) = 0\}$$

for the vanishing set of p. A nonzero polynomial p is said to be *irreducible* if $p = qr$ with $q, r \in \mathbb{C}[z_1, z_2]$ implies that $q \in \mathbb{C}$ or $r \in \mathbb{C}$. .

In the classical Hardy space $H^2(\mathbb{D})$, a function f is called *cyclic* if the smallest closed, shift-invariant subspace generated by its polynomial multiples coincides with the whole space. By Beurling's theorem, cyclic functions in this setting are precisely the outer functions, i.e., $f(0) \neq 0$ and

$$\log|f(0)| = \int_0^{2\pi} \log|f(e^{i\theta})| \frac{d\theta}{2\pi},$$

making the theory both elegant and complete. In contrast, for the Hardy space on the bidisk $H^2(\mathbb{D}^2)$, cyclic functions are outer but there exists an outer function which is not cyclic [17]. With the present understanding, a

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characterization of cyclic function seems to be a harder problem in several variables. However, cyclic polynomials are characterized in $H^2(\mathbb{D}^n)$; they are precisely those polynomials which do not have zeros on the polydisk $\mathbb{D}^n$ [16]. This gap between the univariate and multivariate settings motivates investigations of cyclicity in other spaces, such as the Dirichlet space and the Dirichlet-type spaces.

In the Dirichlet space D of the unit disk, Brown and Shields conjectured [5, Question 12] that a function $f \in D$ is cyclic if and only if it is outer and its boundary zero set has logarithmic capacity zero. They were able to establish the forward direction, while the converse, despite several attempts, remains open till now. Several partial results are known [10, 11]. The Brown–Shields conjecture continues to be a central open problem, motivating further exploration of cyclicity in Dirichlet-type spaces [1, 2, 3, 15].

1.1. **Dirichlet type spaces.** We now introduce the following Dirichlet-type space of the unit bidisk where we will investigate the cyclicity of polynomials. For $\alpha \in \mathbb{R}$, the Dirichlet-type space denoted by $\mathcal{D}_\alpha$ on $\mathbb{D}^2$, consists of all holomorphic functions $f(z,w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\|f\|_\alpha^2 := \sum_{k,l=0}^{\infty} (k+l+1)^\alpha |a_{kl}|^2 < \infty.$$

Note that for $\alpha = 0$, we recover the Hardy space $H^2(\mathbb{D}^2)$ of the unit bidisk and for $\alpha = 1$, the space $\mathcal{D}_1$ was introduced in [4] in connection with toral 2-isometries.

Investigations in this paper are motivated by the cyclicity results [2, 3] obtained for the Dirichlet-type space $\mathfrak{D}_\alpha$, $\alpha \in \mathbb{R}$, on $\mathbb{D}^2$, which consists of holomorphic functions $f(z,w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ such that

$$\|f\|_{\mathfrak{D}_\alpha}^2 := \sum_{k,l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{kl}|^2 < \infty.$$

From the norm definitions, it is straightforward to see that for $\alpha \geq 0$ we have

$$\|\cdot\|_\alpha \leq \|\cdot\|_{\mathfrak{D}_\alpha} \leq \|\cdot\|_{2\alpha} \quad (\mathcal{D}_{2\alpha} \subseteq \mathfrak{D}_\alpha \subseteq \mathcal{D}_\alpha), \tag{1.1}$$

and for $\alpha \leq 0$,

$$\|\cdot\|_{2\alpha} \leq \|\cdot\|_{\mathfrak{D}_\alpha} \leq \|\cdot\|_\alpha \quad (\mathcal{D}_\alpha \subseteq \mathfrak{D}_\alpha \subseteq \mathcal{D}_{2\alpha}). \tag{1.2}$$

Inequalities (1.1), (1.2) allow us to transfer properties between both kinds of Dirichlet-type spaces. In particular, we focus here on the notion of cyclicity.

Note that both spaces $\mathcal{D}_\alpha$ and $\mathfrak{D}_\alpha$ can be viewed as a generalization of the univariate Dirichlet space $D_\alpha, \alpha \in \mathbb{R}$, which consists of holomorphic function $f : \mathbb{D} \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, such that

$$\|f\|_{D_\alpha}^2 := \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty. \tag{1.3}$$

1.2. **Cyclic functions.** In this work, our focus lies on the natural pair of shift operators (M_{z_1}, M_{z_2}) acting on Dirichlet-type spaces $\mathcal{D}_\alpha$. These operators are defined by

$$(M_{z_1}f)(z_1,z_2) = z_1 f(z_1,z_2), \qquad (M_{z_2}f)(z_1,z_2) = z_2 f(z_1,z_2), \quad f \in \mathcal{D}_\alpha.$$

It is straightforward to verify that both M_{z_1} and M_{z_2} are bounded linear operators on $\mathcal{D}_\alpha$. From the operator-theoretic point of view, an important problem is to describe the closed subspaces of $\mathcal{D}_\alpha$ that are invariant under these shifts, namely those $\mathcal{M} \subseteq \mathcal{D}_\alpha$ for which

$$M_{z_1}\mathcal{M} \subseteq \mathcal{M} \qquad \text{and} \qquad M_{z_2}\mathcal{M} \subseteq \mathcal{M}.$$

A key step towards this description is to understand when a function $f \in \mathcal{D}_\alpha$ is cyclic, i.e., when the closed linear span

$$[f] := \overline{\operatorname{span}}\{z_1^k z_2^\ell f : k, \ell \geq 0\}$$

coincides with the entire space $\mathcal{D}_\alpha$. It is clear from the definition that $[f]$ is the smallest closed subspace that contains f and is invariant under the shift operators M_{z_1} and M_{z_2}. Clearly, at least one cyclic vector always exists, e.g., the constant function $f(z_1,z_2) \equiv 1$ is cyclic, because polynomials in two variables are dense in $\mathcal{D}_\alpha$. In the next section, we will see that a necessary condition for cyclicity is that f has no zeros in $\mathbb{D}^2$.

Note that if $g \in [f]$, then $[g] \subseteq [f]$. Thus to check f is cyclic in $\mathcal{D}_\alpha$, it suffices to show that there exists a sequence of polynomials $p_n \in \mathbb{C}[z_1,z_2]$ such that

$$\|p_n f - 1\|_\alpha \to 0 \quad \text{as } n \to \infty.$$

1.3. **Cyclic polynomials in $\mathfrak{D}_\alpha$.** Recent work of Bénéteau et al. [3, Theorem] provides a complete characterization of cyclic polynomials in $\mathfrak{D}_\alpha, \alpha \in \mathbb{R}$, on the bidisk (see [15] for anisotropic setting and [9] for the unit ball in $\mathbb{C}^2$). Their main result shows that the cyclicity of an irreducible polynomial depends intricately on the structure of its zero set on the distinguished boundary $\mathbb{T}^2$. In particular, while non-vanishing in the bidisk is necessary for cyclicity, additional restrictions on the boundary zero set become decisive when the parameter α of the Dirichlet-type space $\mathfrak{D}_\alpha$ lies in the range $(\frac{1}{2}, \infty)$. We recall the result for the reader's convenience.

Theorem 1.1 ([3, Theorem]). *Let $p \in \mathbb{C}[z_1,z_2]$ be an irreducible polynomial with no zeros in the bidisk. We have the following:*

(i) If $\alpha \leq \frac{1}{2}$, then p is cyclic in $\mathfrak{D}_\alpha$.

(ii) If $\frac{1}{2} < \alpha \leq 1$, then p is cyclic in $\mathfrak{D}_\alpha$ if and only if $\mathbb{Z}(p) \cap \mathbb{T}^2$ is an empty or finite set or p is a constant multiple of $\zeta - z_1$ or of $\zeta - z_2$ for some $\zeta \in \mathbb{T}$.

(iii) If $\alpha > 1$, then p is cyclic in $\mathfrak{D}_\alpha$ if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty.

1.4. **Cyclic polynomials in $\mathcal{D}_\alpha$.** In this paper, we solve the problem of characterizing cyclic polynomials in $\mathcal{D}_\alpha$. Our main result is an analog of Theorem 1.1 for the Dirichlet-type space $\mathcal{D}_\alpha$:

Theorem 1.2. *Let $p \in \mathbb{C}[z_1,z_2]$ be an irreducible polynomial with no zeros in the bidisk. We have the following.*

(i) If $\alpha \leq 1$, then p is cyclic in $\mathcal{D}_\alpha$.

(ii) If $1 < \alpha \leq 2$, then p is cyclic in $\mathcal{D}_\alpha$ if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty or finite.

(iii) If $\alpha > 2$, then p is cyclic in $\mathcal{D}_\alpha$ if and only if $\mathcal{Z}(p) \cap \mathbb{T}^2$ is empty.

Remark 1.3. (i) Note that the Dirichlet-type spaces studied in [3] do not distinguish between the cyclicity of a polynomial having finitely many zeros on $\mathbb{T}^2$ and that of $1 - z_i$, $i = 1, 2$, which has infinitely many zeros on $\mathbb{T}^2$ (see Theorem 1.1.(ii)). In fact, the latter case was used there to establish cyclicity for polynomials having finitely many zeros on $\mathbb{T}^2$. In contrast, in our setting, these two cases exhibit different behaviour in $\mathcal{D}_\alpha$.

(ii) Theorem 1.2.(iii) follows directly from Theorem 1.1.(iii) together with the first inclusion in (1.1). In contrast, establishing parts (i) and (ii) of Theorem 1.1 is less straightforward, and our proofs lean on ideas developed in [3, 15]. In particular, the proof of part (i) is obtained by a careful examination of the proof of Theorem 1.1.(ii), with several steps appropriately modified. The proof of part (ii) relies on the techniques used in the proof of [3, Theorem 3.1], together with arguments from [15, Appendix A].

1.5. **Organization of the paper.** In Section 2, we recall some definitions and necessary results to prove our main result. We also present a few necessary conditions for the cyclicity of a function in $\mathcal{D}_\alpha, \alpha \in \mathbb{R}$. In Section 3, we present a proof of Theorem 1.2. This is divided into three subsections. In Subsection 3.1, we consider the spaces $\mathcal{D}_\alpha$ for $\alpha \leq 1$, where we show that the necessary condition for cyclicity – that the polynomial does not vanish on the bidisk – is also sufficient. In Subsection 3.2, we address the case $\alpha \leq 2$. In this case, we show that the known necessary condition for cyclicity stated above is not sufficient, but the polynomial must have at most finitely many zeros on $\mathbb{T}^2$. In Subsection 3.3, we complete the proof of Theorem 1.2 using the results of the preceding sections. In Section 4, we conclude the paper with a discussion of the cyclicity of $1 - z_1 z_2$ and a result based on capacity.

2. PRELIMINARIES

In this section, we list some properties of cyclic function which are needed to give a self-contained treatment of the proof of Theorem 1.2 (see Subsection 2.1), establish two simple criteria to determine cyclicity of a function (see Subsections 2.2 and 2.3) and recall an inequality on the value of a real analytic function (see Subsection 2.4).

2.1. **Some properties of cyclic functions in $\mathcal{D}_\alpha$.** Note that $\mathcal{D}_\alpha, \alpha \in \mathbb{R}$, is a reproducing kernel Hilbert space, i.e., the evaluation map e_w, for $w \in \mathbb{D}$, $e_w(f) := f(w)$ is a continuous linear functional on $\mathcal{D}_\alpha$ and its multiplier space is defined as

$$M(\mathcal{D}_\alpha) = \{\phi : \mathbb{D}^2 \to \mathbb{C} : \phi f \in \mathcal{D}_\alpha \text{ for all } f \in \mathcal{D}_\alpha\}.$$

Elements of $M(\mathcal{D}_\alpha)$ are called *multipliers* of $\mathcal{D}_\alpha$. It is easy to verify that all polynomials are multipliers of $\mathcal{D}_\alpha, \alpha \in \mathbb{R}$.

Some properties of cyclic functions are the following:

(i) A cyclic function in $\mathcal{D}_\alpha$ can not vanish on the bidisk. To see this, take p_n to be a sequence of polynomials such that $\|p_n f - 1\|_\alpha \to 0$. Since evaluations are continuous, the conclusion follows from the following expression

$$p_n(z_1, z_2)f(z_1, z_2) - 1 = e_{(z_1,z_2)}(p_n f - 1).$$

(ii) Given a cyclic function $f \in \mathcal{D}_\alpha, \alpha \in \mathbb{R}$, the function defined by $g(z_1, z_2) := f(\zeta z_1, \eta z_2)$, where $\zeta, \eta \in \mathbb{T}$, is clearly also cyclic in $\mathcal{D}_\alpha$. Indeed, if there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{Z}_+}$ such that

$$\|p_n f - 1\|_\alpha \to 0, \tag{2.1}$$

then the sequence defined by $q_n(z_1, z_2) := p_n(\zeta z_1, \eta z_2)$ satisfies

$$\|q_n g - 1\|_\alpha = \|p_n f - 1\|_\alpha \to 0,$$

proving cyclicity of g.

(iii) Assume that f is a reducible polynomial with $f = gh$ for some nonconstant polynomials g and h. Then $f = gh$ is cyclic in $\mathcal{D}_\alpha$ if and only if g and h are cyclic in $\mathcal{D}_\alpha$. Let us verify:

If f is cyclic, then there is a sequence of polynomials $\{p_n\}_{n\in\mathbb{Z}_+}$ satisfying (2.1). But then the sequence $\{r_n\}_{n\in\mathbb{Z}_+}$ and $\{s_n\}_{n\in\mathbb{Z}_+}$, where $r_n := p_n g$ and $s_n := p_n h$, satisfy $\|r_n h - 1\|_\alpha \to 0$ and $\|s_n g - 1\|_\alpha \to 0$, proving cyclicity of g and h.

Conversely, assume that g and h are cyclic. Then there exists a sequence of polynomials $\{r_n\}_{n\in\mathbb{Z}_+}$ such that $\|r_n g - 1\|_\alpha \to 0$. Note that

$$\|r_n gh - h\|_\alpha \leq \|M_h\| \|r_n g - 1\|_\alpha$$

where $\|M_h\|$ is an operator norm of the multiplication operator M_h. This shows that $h \in [f]$. Hence $[h] \subseteq [f]$. Since h is cyclic, it follows that f is cyclic.

Therefore, it suffices to characterize cyclicity of irreducible polynomials in $\mathcal{D}_\alpha$.

2.2. **Slices of a function.** In this subsection, we establish a result on cyclicity of univariate slices of a function, which serves as quick tools for determining whether a function is a suitable candidate for being cyclic.

Let $f = f(z_1, z_2)$ be a holomorphic function on the bidisk. By fixing one variable, say z_1, the slice

$$f_{z_1} : \mathbb{D} \to \mathbb{C}, \quad f_{z_1}(z_2) \ = \ f(z_1, z_2),$$

is a holomorphic function on the unit disk. The slice f_{z_2} is defined analogously.

Proposition 2.1. *If f is cyclic in $\mathcal{D}_\alpha$, then f_{z_1} and f_{z_2} are cyclic in D_α.*

Proof. Let $\alpha \geq 0$ and $z_1 \in \mathbb{D}$. Consider

$$\begin{aligned}
\|f_{z_1}\|^2_{D_\alpha} \underbrace{=}_{(1.3)} \sum_{j\geq 0}(j+1)^\alpha \left|\sum_{i\geq 0} a_{ij} z_1^i\right|^2 \\
&\leq \sum_{j\geq 0}\left|\sum_{i\geq 0}(i+j+1)^{\alpha/2} a_{ij} z_1^i\right|^2 \\
&\leq \sum_{j\geq 0}\left(\sum_{i\geq 0}(i+j+1)^\alpha |a_{ij}|^2\right)\left(\sum_{i\geq 0}|z_1|^{2i}\right) \\
&\leq \frac{1}{1-|z_1|^2}\|f\|^2_\alpha.
\end{aligned}$$

Let $\alpha \leq 0$. Note that

$$\|f_{z_1}\|_{D_\alpha} \leq \|k_{z_1}\|_{D_\alpha}\|f\|_{\mathfrak{D}_\alpha} \underbrace{\leq}_{(1.2)} \|k_{z_1}\|_{D_\alpha}\|f\|_{\mathcal{D}_\alpha}$$

where k_{z_1} is the reproducing kernel of D_α at z_1 (see [2, Proposition 2.1]). □

A natural question is whether the converse of the above result holds. To examine this, consider $p(z_1, z_2) = 2 - z_1 - z_2$. Note that the slices of p are cyclic in D_α for all α, but p itself is not cyclic in $\mathcal{D}_\alpha$ for $\alpha > 2$ (by Theorem 1.2).

2.3. **Diagonal restriction of a function.** In this subsection, we establish a result on cyclicity of diagonal restriction of a function, which is another criterion for determining whether a function is a suitable candidate for being cyclic.

Given a holomorphic function f on $\mathbb{D}^2$, define the diagonal restriction of f by $(\mathcal{O}f)(z) := f(z, z)$, being a holomorphic function on $\mathbb{D}$. The following proposition gives a necessary condition for the cyclicity of a function in $\mathcal{D}_\alpha$. The proof follows closely the proof of [2, Proposition 2.2].

Proposition 2.2. *If f is cyclic in $\mathcal{D}_\alpha$, then $\mathcal{O}f$ is cyclic in $D_{\alpha-1}$.*

Proof. It suffices to prove that for $f \in \mathcal{D}_\alpha, \alpha \in \mathbb{R}$, it follows that

$$\|\mathcal{O}f\|_{D_{\alpha-1}} \leq \|f\|_\alpha. \tag{2.2}$$

Let $f(z_1, z_2) = \sum_{k=0}^\infty \sum_{l=0}^\infty a_{k,l} z_1^k z_2^l$. Then

$$\mathcal{O}f(z) = \sum_{k=0}^\infty \sum_{l=0}^\infty a_{k,l} z^{k+l}$$

converges absolutely for every $|z| < 1$, hence $\mathcal{O}f$ can be rewritten as

$$\mathcal{O}f(z) = \sum_{n=0}^\infty b_n z^n, \qquad b_n = \sum_{k+l=n} a_{k,l} = \sum_{k=0}^n a_{k,n-k}.$$

Thus,

$$\|\mathcal{O}f\|^2_{D_\alpha} = \sum_{n=0}^{\infty} |b_n|^2 (n+1)^\alpha = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} a_{k,n-k} \right|^2 (n+1)^\alpha. \tag{2.3}$$

By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \left| \sum_{k=0}^{n} a_{k,n-k} \right|^2 &\leq \left(\sum_{k=0}^{n} |a_{k,n-k}|^2 (n+1)^\alpha \right) \left(\sum_{k=0}^{n} (n+1)^{-\alpha} \right) \\ &\leq \sum_{k=0}^{n} |a_{k,n-k}|^2 (n+1)^\alpha (n+1)^{-\alpha+1}. \end{aligned}$$

Using this in (2.3), it follows that

$$\|\mathcal{O}f\|^2_{D_{\alpha-1}} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_{k,n-k}|^2 (n+1)^\alpha = \|f\|^2_\alpha,$$

proving (2.2). □

Remark 2.3. The diagonal restriction in $\mathfrak{D}_\alpha$ satisfies (2.2) only for $\alpha \geq 0$, while for $\alpha < 0$, $\alpha - 1$ needs to be replaced by $2\alpha - 1$ (see [2, Proposition 2.2]).

2.4. **Łojasiewicz's inequality.** In the proof of Theorem 3.3 below, the following inequality will be used essentially.

Theorem 2.4 ([14, Łojasiewicz's inequality])**.** *Let f be a nonzero real analytic function on an open set $U \subseteq \mathbb{R}^n$. Assume the zero set $\mathcal{Z}(f)$ of f in U is nonempty. Let E be a compact subset of U. Then there are constants $C > 0$ and $q \in \mathbb{N}$, depending on E, such that*

$$|f(x)| \geq C \cdot \operatorname{dist}(x, \mathcal{Z}(f))^q$$

for every $x \in E$.

3. Proof of Theorem 1.2

In this section, we provide a proof of Theorem 1.2. In Subsection 3.1, we treat separately the case $\alpha \leq 1$. In Subsection 3.2, we study polynomials having finitely many zeroes on $\mathbb{T}^2$. Finally, in Subsection 3.3, we complete the proof of Theorem 1.2.

3.1. **Polynomial having no zeros inside the bidisk.** The following theorem establishes the cyclicity of polynomials having no zeros in the bidisk. The proof closely parallels the argument of [3, Theorem 4.1] which deals with the cyclicity in $\mathfrak{D}_\alpha$, with a few modifications needed in the framework of $\mathcal{D}_\alpha$.

Theorem 3.1. *Assume that $\alpha \leq 1$. Any polynomial $f \in \mathbb{C}[z_1, z_2]$ that does not vanish in the bidisk is cyclic in $\mathcal{D}_\alpha$.*

Note that it suffices to prove the statement of Theorem 3.1 for $\alpha = 1$. Let us outline the modifications in the proof of [3, Theorem 4.1] to establish Theorem 3.1:

(i) A common property of the spaces $\mathcal{D}_\alpha$ and $\mathfrak{D}_\alpha$ is the *orthogonality of monomials*. Thanks to this property, most of the computational steps in the proof of Theorem 3.1 proceed exactly as in the proof of [3, Theorem 4.1]. The only modification is that the bound $(k+1)d/2+1$ must be replaced by $(k+1)d$ when estimating $\|\vec{v}A^{kd}\vec{B}\|_1$ from above in (3.7).

(ii) Proving Theorem 3.1 for univariate polynomials of the form $z_i - a$, with $|a| \geq 1$, must be done separately, since this case is later used in the proof for general f. This is carried out in Proposition 3.2 below.

For completeness, the full proof of Theorem 3.1 is presented below.

In [5, Lemma 8], Brown and Shields showed that $z-a$ is cyclic in D_1 if and only if $|a| \geq 1$. Since the only candidate for cyclic irreducible polynomial, which depends on only one variable, is $z_i - a, |a| \geq 1$, we study cyclicity of this polynomial in $\mathcal{D}_\alpha, \alpha \in \mathbb{R}$, in the next result.

Proposition 3.2. *Let $p(z) = z - a$ with $|a| \geq 1$. Then $P(z_1, z_2) := p(z_1)$ is cyclic in $\mathcal{D}_\alpha$ if and only if $\alpha \leq 1$.*

Proof. For $f \in D_\alpha$, define $F(z_1, z_2) = f(z_1)$. Note that $\|f\|_{D_\alpha} = \|F\|_{\mathcal{D}_\alpha}$. Since it is well-known [5] that p is cyclic in D_α if and only if $\alpha \leq 1$, the implication ($\Leftarrow$) is clear. For the implication ($\Rightarrow$), assume that $\alpha > 1$. Let $q_n \in \mathbb{C}[z_1, z_2]$ be a sequence such that $\|q_n F - 1\|_\alpha \to 0$. By the orthogonality of monomials in $\mathcal{D}_\alpha$, one can choose $q_n(z_1, z_2) =: g_n(z_1)$ where $g_n \in \mathbb{C}[z]$. But since $\|q_n F - 1\|_\alpha = \|g_n p - 1\|_{D_\alpha}$, this is a contradiction with $\alpha > 1$. $\square$

Now we give a proof of Theorem 3.1.

Proof of Theorem 3.1. We start with a few reductions:

- If f has finitely many or no zeros on $\mathbb{T}^2$, then it is cyclic in $\mathcal{D}_2$ by Theorem 1.2.(ii) and by (1.1), also in $\mathcal{D}_\alpha$ for $\alpha \leq 2$. Thus, to complete the proof of Theorem 3.1, we can assume that f has infinitely many zeros on $\mathbb{T}^2$.
- By (1.1), it suffices to prove Theorem 3.1 for $\alpha = 1$, which we assume from now on and write $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_1$.
- We can assume that f is irreducible (see Subsection 2.1).
- Since f has infinitely many zeros on $\mathbb{T}^2$, it is of the form $f = \lambda \tilde{f}$ for some $\lambda \in \mathbb{T}$, where (n, m) is the bidegree of f and $\tilde{f}(z, w) := z_1^n z_2^m \overline{f\left(\frac{1}{\bar{z}_1}, \frac{1}{\bar{z}_2}\right)}$. This is due to Bézout theorem, because f and $\tilde{f}$ have a common non-constant factor and are irreducible.

Let I_n stand for the $n \times n$ identity matrix. By the reductions above and using [3, Proposition 4.2], there exist a unitary matrix U of size $n+m$, a column vector polynomial $\vec{B}(z_1, z_2) \in \mathbb{C}^{n+m}[z_1, z_2]$ and a row vector polynomial $\vec{v} \in \mathbb{C}^{n+m}[z_2]$, such that

$$\big(I_{n+m} - \underbrace{U(z_1 I_n \oplus z_2 I_m)}_{=:A(z_1,z_2)}\big)\vec{B}(z_1, z_2) \in f(z_1, z_2) \cdot \mathbb{C}^{n+m}[z_1, z_2] \tag{3.1}$$

and

$$p(z_2) := \vec{v}(z_2)\, \vec{B}(z_1, z_2) \in \mathbb{C}[z_2] \quad \text{satisfies} \quad \mathcal{Z}(p) \cap \mathbb{D} = \emptyset. \tag{3.2}$$

Suppose $g \in \mathcal{D}_1$ is orthogonal to $[f]$. Our goal is to show $g = 0$. For this aim we first prove the following claim:

Claim: For every row vector polynomial $\vec{u} \in \mathbb{C}^{n+m}[z_1, z_2]$, it holds that

$$\langle \vec{u}\vec{B}, g\rangle = 0.$$

Proof of Claim. For every $k \in \mathbb{N}$ we have

$$\begin{aligned} (I - A(z_1,z_2)^k)\vec{B}(z_1,z_2) &= \sum_{j=0}^{k-1} A(z_1,z_2)^j (I - A(z_1,z_2))\vec{B}(z_1,z_2) \\ &\underbrace{\in}_{(3.1)} f(z_1,z_2) \cdot \mathbb{C}^{n+m}[z_1,z_2]. \end{aligned} \tag{3.3}$$

For every row vector polynomial $\vec{u} \in \mathbb{C}^{n+m}[z_1, z_2]$ and every $k \in \mathbb{N}$, we have

$$s_k(z_1,z_2) := \vec{u}(z_1,z_2)(I - A(z_1,z_2)^k)\vec{B}(z_1,z_2) \underbrace{\in}_{(3.3)} [f]. \tag{3.4}$$

Since $g \perp [f]$, (3.4) in particular implies that $s_k \perp g$, or equivalently

$$\langle \vec{u}\vec{B}, g\rangle = \langle \vec{u}A^k\vec{B}, g\rangle. \tag{3.5}$$

If $\langle \vec{u}A^k\vec{B}, g\rangle = 0$ for some $k \in \mathbb{N}$, then (3.5) implies the statement of the Claim. Assume that $\langle \vec{u}A^k\vec{B}, g\rangle \neq 0$ for every $k \in \mathbb{N}$. Let

$$d := \deg(\vec{u}(z_1,z_2)) + \deg(\vec{B}(z_1,z_2)) + 1,$$

where $\deg(\cdot)$ denotes the maximum of total degrees of all entries. Since $\vec{u}A^{kd}\vec{B}$ is a linear combination of monomials of degrees between kd and $kd + d - 1$, it follows that $\{\vec{u}A^{kd}\vec{B}\}_{k\geq 0}$ are pairwise orthogonal in $\mathcal{D}_\alpha$.

By Bessel's inequality and (3.5),

$$\|g\|_1^2 \;\geq\; \sum_{k\geq 0} \frac{|\langle \vec{u}\vec{B}, g\rangle|^2}{\|\vec{u}A^{kd}\vec{B}\|_1^2}. \tag{3.6}$$

Further on,

$$\|\vec{u}A^{kd}\vec{B}\|_1^2 \leq (k+1)d\,\|\vec{u}A^{kd}\vec{B}\|_{H^2}^2 \leq (k+1)\underbrace{d\|u\|_{H^\infty}\|^2\vec{B}\|_{H^2}^2}_{=:C} \tag{3.7}$$

where the first inequality follows from the fact that $\vec{u}A^{kd}\vec{B}$ has degree at most $(k+1)d-1$, while in the second inequality we used that $\|A^{kd}(z_1,z_2)\|_{H^\infty} = 1$ for $z_1, z_2 \in \mathbb{T}^2$. Using (3.7) in (3.6), it follows that

$$\sum_{k\geq 0} \frac{|\langle \vec{u}\vec{B}, g\rangle|^2}{C(k+1)} < \infty,$$

which is only possible if $\langle \vec{u}\vec{B}, g\rangle = 0$, proving the Claim. ∎

Choosing $\vec{u}(z_1,z_2) = z_1^j z_2^k \vec{v}(z_2)$, where $\vec{v}$ is as in (3.2), the Claim implies that

$$0 = \langle z_1^j z_2^k \vec{v}(z_2)\vec{B}(z), g\rangle = \langle z_1^j z_2^k p(z_2), g\rangle,$$

whence g is orthogonal to $[p]$. Since p has no zeros in $\mathbb{D}$, it is cyclic in $\mathcal{D}_1$ by Proposition 3.2 and hence $[p] = \mathcal{D}_1$. Therefore $g = 0$ and f is cyclic in $\mathcal{D}_\alpha$ for all $\alpha \leq 1$. □

3.2. **Polynomials having finitely many zeroes on $\mathbb{T}^2$.** In this subsection, we investigate the cyclicity of polynomials in $\mathcal{D}_\alpha$, $\alpha \leq 2$, that possess only finitely many zeros on $\mathbb{T}^2$. In the previously studied settings (see [3, Section 3]), the cyclicity of the polynomial $1 - z_i$ was essentially used in the proof of the main result [3, Theorem 3.1]. However, in our framework, $1 - z_i$ needs not be replaced by another polynomial cyclic in $\mathcal{D}_\alpha$, $\alpha \leq 2$, since $1 - z_i$ is cyclic in $\mathcal{D}_\alpha$ only for $\alpha \leq 1$. In turns out that $2 - z_1 - z_2$ serves as a suitable substitute.

Theorem 3.3. *Consider a polynomial $p \in \mathbb{C}[z_1, z_2]$ having no zeros in $\mathbb{D}^2$ and finitely many on $\mathbb{T}^2$. Then p is cyclic in $\mathcal{D}_\alpha$ for $\alpha \leq 2$.*

The proof of Theorem 3.3 will parallel the arguments in the proof of [3, Theorem 3.2], which deals with the cyclicity in $\mathfrak{D}_\alpha$, using the modifications described in the paragraph before Theorem 3.3. For this reason we first establish Theorem 3.3 in the special case when $p(z_1, z_2) = 2 - z_1 - z_2$. The proof presented here for this special case is motivated by the argument given in [15, Appendix A].

Lemma 3.4. *Let $\alpha \leq 2$. Then $p(z_1, z_2) = 2 - z_1 - z_2$ is cyclic in $\mathcal{D}_\alpha$.*

Proof. It suffices to show that p is cyclic in $\mathcal{D}_2$. Let $f \in \mathcal{D}_2$ be such that $f \perp [p]$. Consider the following series of f,

$$f(z_1, z_2) = \sum_{i,j=0}^{\infty} \frac{b_{i,j}}{(i + j + 1)^2} z_1^i z_2^j.$$

We will show that $f = 0$, or equivalently $b_{i,j} = 0$ for $i, j \in \mathbb{Z}_+$. By $f \perp [p]$, it follows that

$$2b_{k,l} = b_{k+1,l} + b_{k,l+1}, \quad k, l \in \mathbb{Z}_+. \tag{3.8}$$

Since $f \in \mathcal{D}_2$, a new function

$$g(z_1, z_2) := \sum_{i,j \geq 0} b_{i,j} z_1^i z_2^j$$

belongs to $\mathcal{D}_{-2}$, and by (3.8),

$$(z_1 + z_2 - 2z_1 z_2) g(z_1, z_2) = z_1 g(z_1, 0) + z_2 g(0, z_2), \quad (z_1, z_2) \in \mathbb{D}^2. \tag{3.9}$$

We next introduce the substitutions

$$z_1 = \frac{\zeta}{\zeta - 1}, \qquad z_2 = \frac{\zeta}{\zeta + 1}. \tag{3.10}$$

Observe that $z_1 \in \mathbb{D}$ precisely when $\Re\zeta < \frac{1}{2}$, and $z_2 \in \mathbb{D}$ precisely when $\Re\zeta > -\frac{1}{2}$, where $\Re\zeta$ denotes the real part of the complex number ζ. By substituting the above expressions for z_1 and z_2 into (3.9), we get

$$0 = \frac{\zeta}{\zeta - 1} g\left(\frac{\zeta}{\zeta - 1}, 0\right) + \frac{\zeta}{\zeta + 1} g\left(0, \frac{\zeta}{\zeta + 1}\right), \qquad \text{for } -\tfrac{1}{2} < \Re\zeta < \tfrac{1}{2}.$$

Thus, defining $h : \mathbb{C} \to \mathbb{C}$ by

$$h(\zeta) = \begin{cases} \dfrac{1}{\zeta - 1}\, g\left(\dfrac{\zeta}{\zeta-1}, 0\right), & \text{if } \Re\zeta < \frac{1}{2}, \\ -\dfrac{1}{\zeta+1}\, g\left(0, \dfrac{\zeta}{\zeta+1}\right), & \text{if } \Re\zeta > -\frac{1}{2}, \end{cases}$$

we obtain that h is a well-defined entire function. Note that

$$\sum_{k\geq 0} \frac{|b_{k0}|^2}{(k+1)^2} \asymp \int_{\mathbb{D}} |g(z_1, 0)|^2 (1 - |z_1|^2)\, dA(z_1).$$

This can be verified by integrating the right hand side using polar coordinates. Thus we have

$$\begin{aligned} \sum_{k\geq 0} \frac{|b_{k0}|^2}{(k+1)^2} &\asymp \int_{\Re\zeta < 1/2} |(\zeta-1)h(\zeta)|^2 \left(1 - \left|\frac{\zeta}{\zeta-1}\right|^2\right) \frac{dA(\zeta)}{|\zeta-1|^4} \\ &= \int_{\Re\zeta<1/2} |h(\zeta)|^2 \frac{1-2\Re\zeta}{|\zeta-1|^4}\, dA(\zeta), \end{aligned} \tag{3.11}$$

and similarly,

$$\begin{aligned} \sum_{l\geq 0} \frac{|b_{0l}|^2}{(l+1)^2} &\asymp \int_{\mathbb{D}} |g(0, z_2)|^2 (1 - |z_2|^2)\, dA(z_2) \\ &= \int_{\Re\zeta > -1/2} |(\zeta+1)h(\zeta)|^2 \left(1 - \left|\frac{\zeta}{\zeta+1}\right|^2\right) \frac{dA(\zeta)}{|\zeta+1|^4} \\ &= \int_{\Re\zeta>-1/2} |h(\zeta)|^2 \frac{1+2\Re\zeta}{|\zeta+1|^4}\, dA(\zeta). \end{aligned}$$

Both series are finite due to $g \in \mathcal{D}_{-2}$; hence, the sum of the two integrals is finite, and consequently,

$$\int_{|\zeta|>1} \frac{|h(\zeta)|^2}{|\zeta|^4}\, dA(\zeta) < \infty.$$

This forces h to be a polynomial of degree at most 1, i.e.,

$$h(\zeta) = a(\zeta - 1) + b, \quad a, b \in \mathbb{C}. \tag{3.12}$$

Assume that $a \neq 0$. Using (3.12) in (3.11), we get

$$\int_{\Re\zeta<1/2} \frac{1-2\Re\zeta}{|\zeta-1|^2}\, dA(\zeta) < \infty,$$

which is a contradiction. Hence, $a = 0$. Note that

$$\begin{aligned} g(z_1, 0) &\underbrace{=}_{(3.10)} \frac{1}{z_1 - 1}\, h\left(\frac{z_1}{z_1-1}\right) = \frac{b}{z_1 - 1}, \\ g(0, z_2) &\underbrace{=}_{(3.10)} -\frac{1}{z_2+1}\, h\left(\frac{z_2}{z_2+1}\right) = -\frac{b}{z_2+1}. \end{aligned} \tag{3.13}$$

Using (3.13) in (3.9), we get

$$\begin{aligned}(z_1+z_2-2z_1z_2)g(z_1,z_2) &= \frac{bz_1}{z_1-1}-\frac{bz_2}{z_2+1}, \quad (z_1,z_2)\in\mathbb{D}^2,\\ &= \frac{b(z_1+z_2)}{(z_1-1)(z_2-1)}, \quad (z_1,z_2)\in\mathbb{D}^2.\end{aligned} \tag{3.14}$$

In particular, restricting (3.14) to the curve $z_2=-z_1+cz_1^2$, $c\in\mathbb{R}$, we get

$$g(z_1,-z_1+cz_1^2)=\frac{bc}{\big(c+2-cz_1\big)\,(z_1-1)\,(-z_1+cz_1^2+1)}.$$

Therefore

$$g(0,0)=\lim_{z_1\to 0} g(z_1,-z_1+cz_1^2)=\frac{-bc}{c+2}$$

Since c was arbitrary, this implies $b=0$. Therefore $h=0$ and consequently $g=0$ and $f=0$. □

Remark 3.5. The proof shows that if a sequence of complex numbers $\{b_{k,l}\}_{k,l\in\mathbb{Z}_+}$ satisfies (3.8) and

$$\sum_{k,l\in\mathbb{Z}_+}\frac{|b_{k,l}|^2}{(k+l+1)^2}<\infty,$$

then $b_{k,l}=0$ for all $k,l\in\mathbb{Z}_+$.

The following result is an immediate and noteworthy consequence.

Proposition 3.6. *For $\zeta_1,\zeta_2\in\mathbb{T}$, $p(z_1,z_2)=2-\zeta_1z_1-\zeta_2z_2$ is cyclic in $\mathcal{D}_\alpha$ for $\alpha\le 2$.*

Proof. This follows from Lemma 3.4 together with the fact that cyclicity is preserved under rotation. □

The following result, which is an analog of [3, Lemma 3.3], will allow us to compare polynomials having finitely many zeros on $\mathbb{T}^2$ to the polynomials of the type $2-\zeta z_1-\eta z_2, \zeta,\eta\in\mathbb{T}$, whose cyclicity has already been established in Proposition 3.6.

Lemma 3.7. *Suppose $f\in\mathbb{C}[z_1,z_2]$ has no zeros in $\mathbb{D}^2$ and finitely many on $\mathbb{T}^2$, i.e., $\{(\zeta_j,\eta_j)\in\mathbb{T}^2, j=1,\ldots,k\}$. Then for any integer k there exists sufficiently large N such that the function*

$$Q(z_1,z_2)=\frac{\prod_{i=1}^k(2-\zeta_i^{-1}z_1-\eta_i^{-1}z_2)^N}{f(z_1,z_2)}$$

is k-times differentiable on $\mathbb{T}^2$.

Proof. Let $\{(\zeta_j,\eta_j)\in\mathbb{T}^2$, $j=1,\ldots,k$ be as in the statement of the lemma and define the polynomials

$$p_j(z_1,z_2)=2-\zeta_j{}^{-1}z_1-\eta_j{}^{-1}z_2.$$

Clearly, p_j has only one zero (ζ_j,η_j) on $\mathbb{T}^2$. Note that

$$|p_j(z_1,z_2)|^2\le|z_1-\zeta_j|^2+|z_2-\eta_j|^2+2|(z_1-\zeta_j)(z_2-\eta_j)|. \tag{3.15}$$

Writing $z_1,z_2,\zeta_j,\eta_j\in\mathbb{T}$ as

$$z_1=e^{ix_1},\ z_2=e^{ix_2},\ \zeta_j=e^{iy_{1,j}},\ \zeta_j=e^{iy_{2,j}},$$

where $x_1, x_2, y_{1,j}, y_{2,j} \in [0, 2\pi)$, respectively, (3.15) becomes

$$\begin{aligned}|p_j(e^{ix_1}, e^{ix_2})|^2 \le |e^{ix_1} - e^{iy_{1,j}}|^2 + |e^{ix_2} - e^{iy_{2,j}}|^2 \\ + 2\left|(e^{ix_1} - e^{iy_{1,j}})(e^{ix_2} - e^{iy_{2,j}})\right|.\end{aligned} \tag{3.16}$$

Define

$$r(x_1, x_2) = |f(e^{ix_1}, e^{ix_2})|^2, \qquad (x_1, x_2) \in \mathbb{R}^2,$$

and let $\mathcal{Z}(r)$ denote its zero set. Consider the compact set $E = [0, 2\pi]^2$. By Theorem 2.4, there exist constant $C > 0$ and $q \in \mathbb{N}$ such that

$$r(x) \ge C \operatorname{dist}(x, \mathcal{Z}(r))^q, \qquad x \in E.$$

Since the set $\mathcal{Z}(r) \cap E$ consists of finitely many points, there is a constant $c > 0$ satisfying

$$\operatorname{dist}(x, \mathcal{Z}(r))^2 \ge c \prod_{y \in \mathcal{Z}(r) \cap E} |x - y|^2, \qquad x \in E. \tag{3.17}$$

Also, we have

$$|x - y|^2 = \sum_{j=1,2} |x_j - y_j|^2 \ge \sum_{j=1,2} |e^{ix_j} - e^{iy_j}|^2 \ge 2 \prod_{j=1,2} |e^{ix_j} - e^{iy_j}|. \tag{3.18}$$

Using (3.16),(3.17) and (3.18), we have a constant $C_1 > 0$ such that

$$\operatorname{dist}(x, \mathcal{Z}(r))^2 \ge C_1 \prod_j |p_j|^2.$$

This yields that the function

$$\frac{\prod_j |p_j|^{\,q/2}}{|f(z_1, z_2)|^2},$$

is bounded on $\mathbb{T}^2$. We now apply the standard trick, which is to increase the exponent in the numerator and assign the value zero at the zeros of f, to obtain a function that is k-times continuously differentiable on $\mathbb{T}^2$. This completes the proof of Lemma 3.7. □

Finally, we can prove Theorem 3.3

Proof of Theorem 3.3. By Lemma 3.7, we obtain a function

$$Q(z_1, z_2) = \frac{\left(\prod_{i=1}^k (2 - \zeta_i^{-1} z_1 - \eta_i^{-1} z_2)\right)^N}{p(z_1, z_2)} =: \frac{g(z_1, z_2)}{p(z_1, z_2)},$$

where $(\zeta_i, \eta_i) \in \mathbb{T}^2$, which is twice continuously differentiable on $\mathbb{T}^2$. Thus its Fourier coefficients $\widehat{Q}(k, l)$ satisfy

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k + 1)^2 (l + 1)^2 < \infty.$$

But since

$$\sum_{k,l} |\widehat{Q}(k, l)|^2 (k + l + 1)^2 \le \sum_{k,l} |\widehat{Q}(k, l)|^2 (k + 1)^2 (l + 1)^2,$$

we obtain $Q \in \mathcal{D}_\alpha, \alpha \le 2$. Hence $g(z_1, z_2) \in p\mathcal{D}_\alpha, \alpha \le 2$. Since g is cyclic in $\mathcal{D}_2$ and p is a multiplier, we obtain that p is also cyclic in $\mathcal{D}_2$. □

3.3. **Proof of Theorem 1.2.** The proof of Theorem 1.2 is now straightforward using the results above.

We divide the argument into three cases according to the value of α: $\alpha > 2$, $\alpha \in (1, 2]$ and $\alpha \leq 1$.

Case 1: $\alpha > 2$. Using the norm inequalities $\|f\|_{\mathfrak{D}_{\alpha/2}} \leq \|f\|_{\mathcal{D}_\alpha} \leq \|f\|_{\mathfrak{D}_\alpha}$, Theorem 1.2.$(iii)$ follows from Theorem 1.1.(iii).

Case 2: $\alpha \in (1, 2]$. The norm inequality $\|f\|_\alpha \geq \|f\|_{\mathfrak{D}_{\alpha/2}}$ implies that if a polynomial p is cyclic in $\mathcal{D}_\alpha$, it must also be cyclic in $\mathfrak{D}_{\alpha/2}$. By Theorem 1.1, the zero set $Z(p) \cap \mathbb{T}^2$ is empty or a finite set.

Subcase 2.1: $Z(p) \cap \mathbb{T}^2 = \varnothing$. Using the case $\alpha > 2$, we obtain that p is cyclic in $\mathcal{D}_3$. Since $\|\cdot\|_\alpha \leq \|\cdot\|_3$ for $\alpha \leq 2$, cyclicity in $\mathcal{D}_\alpha$ follows.

Subcase 2.2: $Z(p) \cap \mathbb{T}^2$ is finite. This subcase follows by Theorem 3.3.

Case 3: $\alpha \leq 1$. This subcase follows by Theorem 3.1.

Remark 3.8. Note that if p has no zeros in the bidisk and only finitely many zeros on $\mathbb{T}^2$, then its cyclicity in $\mathcal{D}_\alpha$, $\alpha \leq 1$, also follows from Theorem 1.1 by a direct comparison of norms. However, this result excludes the cyclicity of a polynomial having infinitely many zeros on $\mathbb{T}^2$ as in the case of $1 - z_1 z_2$.

4. Concluding remarks

We provided a self-contained proof of the cyclicity of the polynomial $2 - z_1 - z_2$; we now give a simple proof of the cyclicity of $1 - z_1 z_2$ using standard technique. These are model polynomials and often play a crucial role in establishing cyclicity results for broader classes of functions (i.e., see the proof of [15, Theorem 5]), and therefore merit particular attention to ensure independent proof.

Lemma 4.1. *$1 - z_1 z_2$ is cyclic in $\mathcal{D}_\alpha$ if and only if $\alpha \leq 1$.*

Proof. Note that if f is a function in one variable say z and F is defined by $F(z_1, z_2) := f(z_1 z_2)$, then we have the following: for $\alpha > 0$

$$\|f\|_{D_\alpha} \leq \|F\|_\alpha \leq 2^\alpha \|f\|_{D_\alpha}. \tag{4.1}$$

Let $\alpha \in \mathbb{R}$. Assume that $p(z_1, z_2) = 1 - z_1 z_2$ is cyclic in $\mathcal{D}_\alpha$. Then there exists sequence of polynomials $P_n(z_1, z_2) := p_n(z_1 z_2)$ such that $\|P_n p - 1\|_\alpha \to 0$, where p_n is a univariate polynomial. By (4.1), we have that $1 - z$ is cyclic in D_α and hence $\alpha \leq 1$. The converse also follows using similar reasoning. $\square$

We conclude the paper with a few comments on capacity. Finite logarithmic capacity, or Riesz α-capacity plays an important role in identifying non-cyclic functions in $\mathfrak{D}_\alpha$ [2, Proposition 4.2]. This approach originates in the work of Brown and Shields [5] and was later extended to several variables by several authors. We record the following straightforward result, since it provides a necessary condition for the cyclicity of a general function in $\mathcal{D}_\alpha, \alpha \in \mathbb{R}$.

Proposition 4.2. *Let $\alpha \in \mathbb{R}$, and $f \in \mathcal{D}_\alpha$. Let f^* denote the radial limit of f i.e. $f^*(e^{i\theta_1}, e^{i\theta_2}) = \lim_{r\to 1^-} f(re^{i\theta_1}, re^{i\theta_2})$. Then the following holds:*

(a) *If $\mathcal{Z}(f^*)$ has positive logarithmic capacity, then f is not cyclic in $\mathcal{D}_\alpha, \alpha \geq 2$.*
(b) *For $0 < \alpha < 1$, if $\mathcal{Z}(f^*)$ has positive Riesz α-capacity, then f is not cyclic in $\mathcal{D}_{2\alpha}$.*

Proof. This follows from the fact that f is not cyclic in respective $\mathfrak{D}_\alpha$ and by doing norm comparison with $\mathcal{D}_{2\alpha}$. $\square$

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