TWO-POINT COPULA INTERPOLATION BY COORDINATE PROJECTION

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Abstract. We give a short inductive proof, based on coordinate projection and gluing of probability measures, of the known equivalence between copula and quasi-copula interpolation at two arbitrary points of the unit cube. The admissible values are characterized by the one-point Fréchet–Hoeffding bounds and the two directed difference bounds. Starting from the established bivariate result, we merge two coordinates having the same order at the interpolation points. Explicit formulas for the two merged values preserve all interpolation inequalities. A gluing construction restores the original coordinates and yields a checkerboard copula. Since two equally ordered coordinates exist in every dimension at least three, the argument applies successively in all dimensions.

1. Introduction

Values of a copula at prescribed points are probabilities of joint threshold events and therefore encode partial information about dependence. Before using such information to derive bounds or construct a model, one must determine whether it is compatible with a probability distribution having the required margins. This question connects finite interpolation with best-possible bounds for multivariate distributions [4]. Bounds incorporating prescribed copula values also have applications to model-free pricing of two-asset options [5]. The two-point problem provides a basic setting in which compatibility can be characterized by explicit inequalities.

Bivariate interpolation was studied by Mardani-Fard, Sadooghi-Alvandi, and Shishebor [3]. De Baets, De Meyer, Fernández-Sánchez, and Úbeda-Flores [1] established that two points in dimension three admit a copula interpolant whenever they admit a quasi-copula interpolant, using a linear programming argument. Klement, Kokol Bukovšek, Omladič, Saminger-Platz, and Stopar [2] proved this equivalence in arbitrary dimension by a constructive argument involving distributions with prescribed margins, slicing, and extension.

In this paper we give a short proof of the two-point interpolation result stated in Theorem 2.1, based on induction on the dimension, explicit coordinate projection, and gluing of probability measures. Two coordinate columns having the same order at the interpolation points are replaced by a single column. Explicit formulas for its entries preserve the interpolation inequalities and are realizable by a bivariate copula at the two pairs of original coordinates. A gluing construction restores these coordinates after interpolation in one lower dimension. Since there are only two possible orders of the point labels, the reduction is available in every dimension at least three.

2. Preliminaries

For $d \geq 2$, a *d-copula* is the joint distribution function of d random variables with uniform distributions on $[0,1]$. A *d-quasi-copula* is a grounded function $Q : [0,1]^d \to [0,1]$ with uniform

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one-dimensional margins that is nondecreasing in each coordinate and satisfies, for all points $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_d)$ in $[0,1]^d$,

$$|Q(\mathbf{x}) - Q(\mathbf{y})| \leq \sum_{i=1}^{d} |x_i - y_i|.$$

Here groundedness means $Q(\mathbf{x}) = 0$ whenever some coordinate is zero, and uniform margins mean $Q(1, \ldots, 1, t, 1, \ldots, 1) = t$ in every coordinate. Every copula is a quasi-copula. We write $a_+ = \max\{a, 0\}$.

The following theorem gives the criterion for interpolation at two arbitrary points. The equivalence between copula and quasi-copula interpolation was established for $d = 2$ in [3, Theorem 2.2], for $d = 3$ in [1], and for arbitrary d in [2].

Theorem 2.1. *Let $d \geq 2$ and let $\mathbf{x}^{(1)} = (x_1^{(1)}, \ldots, x_d^{(1)})$ and $\mathbf{x}^{(2)} = (x_1^{(2)}, \ldots, x_d^{(2)})$ be points in $[0,1]^d$. For $q_1, q_2 \in \mathbb{R}$, the following statements are equivalent:*

1. *There exists a d-copula C with $C(\mathbf{x}^{(p)}) = q_p$ for $p = 1, 2$.*
2. *There exists a d-quasi-copula Q with $Q(\mathbf{x}^{(p)}) = q_p$ for $p = 1, 2$.*
3. *The data satisfy*

$$\max\left\{0, \sum_{i=1}^{d} x_i^{(p)} - d + 1\right\} \leq q_p \leq \min_{1 \leq i \leq d} x_i^{(p)} \qquad (p = 1, 2), \tag{2.1}$$

$$q_a - q_b \leq \sum_{i=1}^{d} (x_i^{(a)} - x_i^{(b)})_+ \qquad ((a, b) = (1, 2), (2, 1)). \tag{2.2}$$

When these conditions hold, C can be chosen to be a checkerboard copula on the grid generated by the coordinates of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

The equivalence between assertions (2) and (3) follows from the sharp one-point bounds for quasi-copulas in [4, Theorem 3.2].

Definition 2.2. We call the interpolation data $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, q_1, q_2)$ *admissible*, if it satisfies the conditions (2.1)–(2.2).

2.1. **Checkerboard extension.** The following standard construction will be used throughout. For each coordinate i, take an ordered grid

$$0 = t_{i0} \leq t_{i1} \leq t_{i2} \leq t_{i3} = 1, \qquad \delta_{ij} = t_{ij} - t_{i,j-1} \quad (j = 1, 2, 3).$$

Set $J_{ij} = [t_{i,j-1}, t_{ij})$ and

$$B_a = \prod_{i=1}^{d} J_{i,a_i}, \qquad a \in \{1, 2, 3\}^d.$$

For each i, the half-open convention makes the intervals J_{i1}, J_{i2}, J_{i3} pairwise disjoint; consequently, the cells B_a are pairwise disjoint as well. The endpoint 1 is omitted in each coordinate, but the omitted boundary faces have probability zero for measures with uniform margins. In the grid generated by two points, the ordered coordinates t_{ij} are given by $t_{i1} = \min(x_i^{(1)}, x_i^{(2)})$ and $t_{i2} = \max(x_i^{(1)}, x_i^{(2)})$.

The following lemma constructs a checkerboard copula from cell masses with the required marginal totals.

Lemma 2.3. *Suppose numbers m_a, indexed by $a \in \{1, 2, 3\}^d$, satisfy*

$$m_a \geq 0, \qquad \sum_{a : a_i = j} m_a = \delta_{ij} \quad (i = 1, \ldots, d,\ j = 1, 2, 3). \tag{2.3}$$

Spreading m_a uniformly over each positive-volume cell B_a defines a copula. At any grid node $\mathbf{x}$*, its value is*

$$C(\mathbf{x}) = \sum_{a:\ t_{i,a_i} \le x_i \ \text{for every } i} m_a. \tag{2.4}$$

Conversely, the cell masses of any copula satisfy (2.3)*. Replacing its restriction to each cell by a uniform distribution preserves its values at all grid nodes.*

Proof. Summing the margin equations gives $\sum_a m_a = 1$. Nonnegativity forces $m_a = 0$ whenever one of the side lengths δ_{i,a_i} is zero. For a positive-volume cell, define

$$f(\mathbf{t}) = \frac{m_a}{\prod_{i=1}^d \delta_{i,a_i}} \quad (\mathbf{t} = (t_1, t_2, \dots, t_d) \in B_a).$$

Define $f = 0$ on the remaining null set. Thus no zero side length appears in a denominator. The function f is a probability density. On an interval J_{ij} of positive length, its ith marginal density is

$$f_i(t_i) = \int_{[0,1]^{d-1}} f(\mathbf{t})\, dt_{-i} = \frac{1}{\delta_{ij}} \sum_{a:a_i=j} m_a = 1,$$

Here the integral is with respect to $(d-1)$-dimensional Lebesgue measure on $[0,1]^{d-1}$ in the coordinates other than i, with t_i held fixed; thus $dt_{-i} = dt_1 \cdots dt_{i-1}\, dt_{i+1} \cdots dt_d$. Hence its distribution function is a copula, and summing the cells below a grid node gives (2.4). For the converse, uniform margins give (2.3). Cell boundaries have probability zero, again because the margins are uniform. □

For completeness, the checkerboard copula has the explicit formula

$$C(\mathbf{t}) = \sum_{a:\ \delta_{i,a_i} > 0 \text{ for every } i} m_a \prod_{i=1}^d \min\left\{1, \max\left\{0, \frac{t_i - t_{i,a_i-1}}{\delta_{i,a_i}}\right\}\right\}. \tag{2.5}$$

Every factor is a one-dimensional uniform distribution function on the corresponding coordinate interval.

3. Projection in the original coordinates

Let $d \ge 3$, and let $\mathbf{x}^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots, x_d^{(1)})$ and $\mathbf{x}^{(2)} = (x_1^{(2)}, x_2^{(2)}, \dots, x_d^{(2)})$ be points in $[0,1]^d$ whose first two coordinates have the same order. If needed, one can interchange the point labels, so we can assume $x_1^{(1)} \le x_1^{(2)}$ and $x_2^{(1)} \le x_2^{(2)}$. The next lemma replaces these two coordinates by one while preserving the interpolation conditions.

Lemma 3.1. *Let $d \ge 3$, let*

$$\mathbf{x}^{(p)} = (x_1^{(p)}, x_2^{(p)}, \dots, x_d^{(p)}) \in [0,1]^d \qquad (p = 1, 2),$$

and suppose

$$x_1^{(1)} \le x_1^{(2)}, \qquad x_2^{(1)} \le x_2^{(2)}.$$

Let $q_1, q_2 \in \mathbb{R}$ satisfy (2.1)–(2.2) *(namely $(x_1^1, x_2^1), (x_1^2, x_2^2), q_1, q_2$) is admissible). Define*

$$\begin{aligned} z_1 &= \max\{q_1,\ q_2 - (x_1^{(2)} - x_1^{(1)}) - (x_2^{(2)} - x_2^{(1)}),\ x_1^{(1)} + x_2^{(1)} - 1\}, \\ z_2 &= \max\{q_1,\ q_2,\ x_1^{(2)} + x_2^{(2)} - 1\}. \end{aligned} \tag{3.1}$$

Then $0 \le z_1 \le z_2 \le 1$, and there exist a bivariate checkerboard copula B satisfying

$$B(x_1^{(p)}, x_2^{(p)}) = z_p \qquad (p = 1, 2).$$

Moreover, the reduced points

$$\widetilde{\mathbf{x}}^{(p)} = (z_p, x_3^{(p)}, \dots, x_d^{(p)}) \in [0,1]^{d-1} \qquad (p = 1, 2)$$

with the same prescribed values q_1, q_2 satisfy

$$\max\left\{0, z_p + \sum_{i=3}^{d} x_i^{(p)} - (d-2)\right\} \leq q_p \leq \min\{z_p, x_3^{(p)}, \dots, x_d^{(p)}\} \qquad (p=1,2), \tag{3.2}$$

$$q_1 - q_2 \leq \sum_{i=3}^{d} (x_i^{(1)} - x_i^{(2)})_+, \tag{3.3}$$

$$q_2 - q_1 \leq z_2 - z_1 + \sum_{i=3}^{d} (x_i^{(2)} - x_i^{(1)})_+. \tag{3.4}$$

Thus the reduced data are admissible in dimension $d-1$.

Proof. **The bivariate values z_1, z_2:** The definition (3.1), together with $q_p \geq 0$ from (2.1), gives $z_p \geq \max\{0, x_1^{(p)} + x_2^{(p)} - 1\}$ for $p = 1, 2$. Every term defining z_1 is at most both $x_1^{(1)}$ and $x_2^{(1)}$. For its middle term, the upper bounds $q_2 \leq x_1^{(2)}, x_2^{(2)}$ in (2.1) give

$$\begin{aligned} q_2 - (x_1^{(2)} - x_1^{(1)}) - (x_2^{(2)} - x_2^{(1)}) &\leq x_1^{(1)} - (x_2^{(2)} - x_2^{(1)}) \leq x_1^{(1)}, \\ q_2 - (x_1^{(2)} - x_1^{(1)}) - (x_2^{(2)} - x_2^{(1)}) &\leq x_2^{(1)} - (x_1^{(2)} - x_1^{(1)}) \leq x_2^{(1)}. \end{aligned}$$

The other two terms are controlled by $q_1 \leq \min\{x_1^{(1)}, x_2^{(1)}\}$ from (2.1) and by $x_1^{(1)} + x_2^{(1)} - 1 \leq \min\{x_1^{(1)}, x_2^{(1)}\}$, since the coordinates are at most 1. Applying (2.1) and $x_i^{(1)} \leq x_i^{(2)}$ for $i = 1, 2$ similarly shows that every term defining z_2 is at most both $x_1^{(2)}$ and $x_2^{(2)}$. Consequently

$$\max\{0, x_1^{(p)} + x_2^{(p)} - 1\} \leq z_p \leq \min\{x_1^{(p)}, x_2^{(p)}\} \qquad (p=1,2). \tag{3.5}$$

Termwise comparison in (3.1) gives $z_1 \leq z_2$. Also,

$$\begin{aligned} &z_1 + (x_1^{(2)} - x_1^{(1)}) + (x_2^{(2)} - x_2^{(1)}) \\ &\quad = \max\big\{q_1 + (x_1^{(2)} - x_1^{(1)}) + (x_2^{(2)} - x_2^{(1)}),\ q_2,\ x_1^{(2)} + x_2^{(2)} - 1\big\} \geq z_2. \end{aligned}$$

Hence

$$0 \leq z_2 - z_1 \leq (x_1^{(2)} - x_1^{(1)}) + (x_2^{(2)} - x_2^{(1)}). \tag{3.6}$$

Equations (3.5)–(3.6) are the bivariate interpolation conditions for the two comparable points $(x_1^{(p)}, x_2^{(p)})$ and the values z_p. The constructive result [3, Lemma 2.1 and Theorem 2.2] therefore supplies B.

The reduced one-point bounds for $(z_p, x_3^{(p)}, \dots, x_d^{(p)})$, $p = 1, 2$: We verify (3.2). Nonnegativity of q_p follows from (2.1). The reduced upper bounds follow from $q_p \leq z_p$ in (3.1) and $q_p \leq x_i^{(p)}$ in (2.1) for $i = 3, \dots, d$. For the lower bounds, we shall prove

$$z_p \leq q_p + \sum_{i=3}^{d} (1 - x_i^{(p)}) \qquad (p=1,2). \tag{3.7}$$

For $p = 1$, (2.2) with $(a, b) = (2, 1)$ gives

$$q_2 - (x_1^{(2)} - x_1^{(1)}) - (x_2^{(2)} - x_2^{(1)}) \leq q_1 + \sum_{i=3}^{d} (x_i^{(2)} - x_i^{(1)})_+ \leq q_1 + \sum_{i=3}^{d} (1 - x_i^{(1)}).$$

The lower Fréchet bound in (2.1) for $p = 1$ gives

$$x_1^{(1)} + x_2^{(1)} - 1 \leq q_1 + \sum_{i=3}^{d} (1 - x_i^{(1)}).$$

The remaining term q_1 is bounded by the same expression, so taking the maximum proves (3.7) for $p = 1$. For $p = 2$, (2.2) with $(a, b) = (1, 2)$ and the lower Fréchet bound in (2.1) give

$$q_1 \le q_2 + \sum_{i=3}^{d} (x_i^{(1)} - x_i^{(2)})_+ \le q_2 + \sum_{i=3}^{d} (1 - x_i^{(2)}),$$
$$x_1^{(2)} + x_2^{(2)} - 1 \le q_2 + \sum_{i=3}^{d} (1 - x_i^{(2)}).$$

Together with the trivial bound for the term q_2, these inequalities prove (3.7) for $p = 2$. Rearranging (3.7) yields the lower bound in (3.2).

The reduced directed difference bounds (3.3)–(3.4)**:** Because the first two coordinates are nondecreasing from point 1 to point 2, the condition (2.2) with $(a, b) = (1, 2)$ is already the condition (3.3). To prove (3.4), consider the term attaining the maximum that defines z_1 in (3.1).

Case 1: $z_1 = q_1$. Since $z_2 \ge q_2$ by (3.1), we obtain

$$q_2 - q_1 \le z_2 - z_1 \le z_2 - z_1 + \sum_{i=3}^{d} (x_i^{(2)} - x_i^{(1)})_+.$$

Case 2: $z_1 = q_2 - (x_1^{(2)} - x_1^{(1)}) - (x_2^{(2)} - x_2^{(1)})$. The bound $z_2 \ge q_2$ in (3.1) gives

$$z_2 - z_1 \ge (x_1^{(2)} - x_1^{(1)}) + (x_2^{(2)} - x_2^{(1)}).$$

Combining this with (2.2) for $(a, b) = (2, 1)$ yields

$$q_2 - q_1 \le (x_1^{(2)} - x_1^{(1)}) + (x_2^{(2)} - x_2^{(1)}) + \sum_{i=3}^{d} (x_i^{(2)} - x_i^{(1)})_+$$
$$\le z_2 - z_1 + \sum_{i=3}^{d} (x_i^{(2)} - x_i^{(1)})_+.$$

Case 3: $z_1 = x_1^{(1)} + x_2^{(1)} - 1$. Using $z_2 \ge x_1^{(2)} + x_2^{(2)} - 1$ from (3.1), we again have

$$z_2 - z_1 \ge (x_1^{(2)} - x_1^{(1)}) + (x_2^{(2)} - x_2^{(1)}).$$

The same application of (2.2) as in Case 2 gives (3.4). The three cases cover all possible maximizing terms in (3.1), including equalities between them. □

Remark 3.2. The merged values in (3.1) are chosen to meet the reduced upper bound in (3.2) and the bivariate interpolation requirement in Lemma 3.1. The former requires $z_p \ge q_p$, while the existence of B with $B(x_1^{(p)}, x_2^{(p)}) = z_p$ implies (3.5)–(3.6). In particular,

$$z_p \ge x_1^{(p)} + x_2^{(p)} - 1 \quad (p = 1, 2), \qquad 0 \le z_2 - z_1 \le (x_1^{(2)} - x_1^{(1)}) + (x_2^{(2)} - x_2^{(1)}).$$

The bound $z_2 \ge q_2$ therefore forces

$$z_1 \ge q_2 - (x_1^{(2)} - x_1^{(1)}) - (x_2^{(2)} - x_2^{(1)}),$$

while $z_1 \ge q_1$ forces $z_2 \ge q_1$. These are exactly the terms in (3.1). The lemma shows that taking their maxima also preserves the reduced one-point and directed difference bounds.

4. Gluing and induction in dimension

The projection lemma produces a bivariate copula for the selected pair of coordinates. Once the reduced interpolation problem has been solved, the two probability measures can be joined along their nested threshold events as follows.

Lemma 4.1. *Let $d \geq 3$ and let $\mathbf{x}^{(1)} = (x_1^{(1)}, \ldots, x_d^{(1)})$ and $\mathbf{x}^{(2)} = (x_1^{(2)}, \ldots, x_d^{(2)})$ be points in $[0,1]^d$ such that $x_1^{(1)} \leq x_1^{(2)}$ and $x_2^{(1)} \leq x_2^{(2)}$. Suppose a bivariate copula B and a $(d-1)$-copula D satisfy*

$$B(x_1^{(p)}, x_2^{(p)}) = z_p, \qquad D(z_p, x_3^{(p)}, \ldots, x_d^{(p)}) = q_p \qquad (p = 1, 2).$$

Then:

1. *There exists a d-copula C such that $C(\mathbf{x}^{(p)}) = q_p$ for $p = 1, 2$.*
2. *If B and D are checkerboard copulas on their respective grids generated by these two pairs of points, C can be chosen to be a checkerboard copula on the grid generated by $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$.*

Proof. Let μ_B be the probability measure associated with B. Set

$$R_p = [0, x_1^{(p)}] \times [0, x_2^{(p)}] \quad (p = 1, 2), \qquad H_1 = R_1, \quad H_2 = R_2 \setminus R_1, \quad H_3 = [0,1]^2 \setminus R_2.$$

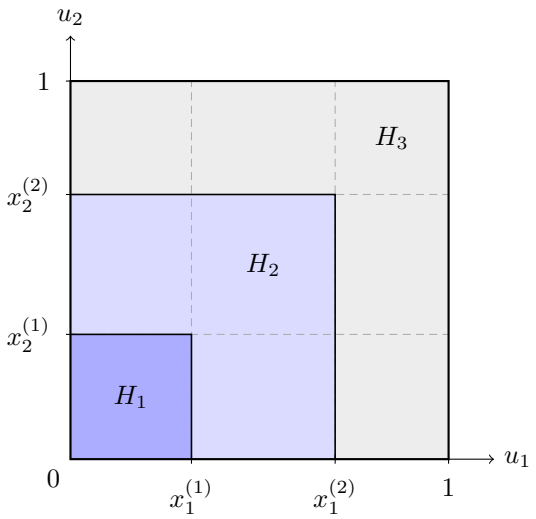


Figure 1. The partition into H_1, H_2, H_3 for strictly ordered interior thresholds. Dashed lines show the coordinate grid; H_2 and H_3 are unions of its cells.

These sets partition the square and have probabilities

$$\ell_1 = z_1, \qquad \ell_2 = z_2 - z_1, \qquad \ell_3 = 1 - z_2. \tag{4.1}$$

Let $(T, Y_3, \ldots, Y_d)$ be a random vector on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with joint distribution function D, where $\mathbb{P}$ denotes the probability measure on this space. Put

$$I_1 = [0, z_1), \qquad I_2 = [z_1, z_2), \qquad I_3 = [z_2, 1].$$

The uniformity of T gives $\mathbb{P}(T \in I_k) = \ell_k$. For Borel sets $E \subseteq [0,1]^2$ and $F \subseteq [0,1]^{d-2}$, define

$$\mu_k(E) = \mu_B(E \cap H_k), \qquad \nu_k(F) = \mathbb{P}\big(T \in I_k, (Y_3, \ldots, Y_d) \in F\big).$$

Both finite measures have total mass ℓ_k. On $[0,1]^2 \times [0,1]^{d-2}$, define

$$\mu = \sum_{k:\ell_k > 0} \frac{\mu_k \otimes \nu_k}{\ell_k}. \tag{4.2}$$

Here $\mu_k \otimes \nu_k$ denotes the product measure, characterized on Borel rectangles by $(\mu_k \otimes \nu_k)(E \times F) = \mu_k(E)\nu_k(F)$. Each summand has mass ℓ_k, so μ is a probability measure. For Borel sets $E \subseteq [0,1]^2$ and $F \subseteq [0,1]^{d-2}$,

$$\mu(E \times [0,1]^{d-2}) = \sum_k \mu_k(E) = \mu_B(E),$$
$$\mu([0,1]^2 \times F) = \sum_k \nu_k(F) = \mathbb{P}\big((Y_3, \dots, Y_d) \in F\big).$$

Thus the joint marginal on the first two coordinates is μ_B. The joint marginal on coordinates $3, \dots, d$, obtained by ignoring the first two coordinates, is the distribution of $(Y_3, \dots, Y_d)$ under D. All one-dimensional margins of μ are therefore uniform.

For $F_p = \prod_{i=3}^d [0, x_i^{(p)}]$, the nesting of R_1, R_2 gives

$$\begin{aligned}\mu(R_p \times F_p) &= \sum_{k=1}^p \nu_k(F_p) \\ &= \mathbb{P}(T \le z_p,\ Y_i \le x_i^{(p)} \text{ for } i = 3, \dots, d) \\ &= D(z_p, x_3^{(p)}, \dots, x_d^{(p)}) = q_p.\end{aligned}$$

Boundary sets have probability zero because the margins are uniform. Thus the distribution function of μ is the required copula.

If B and D are checkerboard copulas on the stated grids, each H_k is a union of bivariate grid cells up to boundaries. The restriction μ_k has constant density on each such cell. Integrating the density of D over I_k gives a measure ν_k with constant density on every remaining grid cell. Formula (4.2) consequently has constant density on every original grid cell. □

Proof of Theorem 2.1. (2) and (3) are equivalent by the sharp quasi-copula bounds in [4, Theorem 3.2]. Every copula is a quasi-copula, so (1) implies (2). It remains to construct a copula from admissible data. We prove the stronger checkerboard assertion by induction on dimension. The case $d = 2$ is the constructive bivariate result [3, Lemma 2.1 and Theorem 2.2].

Assume the assertion holds in dimension $n \ge 2$, and consider two admissible points in dimension $n+1$. Every coordinate belongs to one of the two groups specified by $x_i^{(1)} \le x_i^{(2)}$ and $x_i^{(1)} > x_i^{(2)}$. Since there are at least three coordinates, one group contains at least two indices. Permute coordinates to place two such indices first and, if needed, interchange the point labels. Then $x_1^{(1)} \le x_1^{(2)}$ and $x_2^{(1)} \le x_2^{(2)}$.

Apply Lemma 3.1 in dimension $n+1$. It supplies a bivariate checkerboard copula B and admissible reduced data in dimension n. The induction hypothesis supplies a checkerboard copula D for those reduced data. Lemma 4.1 combines them into an interpolating checkerboard copula in dimension $n+1$. Undoing the coordinate permutation and point relabeling completes the induction. Repeated coordinates and boundary values give zero-mass bands, which are omitted in (4.2). □

The gluing construction has the following explicit form in terms of checkerboard cell masses.

Remark 4.2. Let b_{ij}, $i, j \in \{1,2,3\}$, be the masses of B on the grids $(0, x_1^{(1)}, x_1^{(2)}, 1)$ and $(0, x_2^{(1)}, x_2^{(2)}, 1)$. Let d_{ka}, with $k \in \{1,2,3\}$ and $a \in \{1,2,3\}^{d-2}$, be the masses of D: the first index refers to the intervals with endpoints $0, z_1, z_2, 1$, and a indexes the remaining coordinate intervals. Up to null boundaries, the bivariate cell (i,j) belongs to $H_{\max(i,j)}$. Hence

$$m_{ija} = \begin{cases} \dfrac{b_{ij} d_{\max(i,j),a}}{\ell_{\max(i,j)}}, & \ell_{\max(i,j)} > 0, \\ 0, & \ell_{\max(i,j)} = 0, \end{cases} \qquad i,j \in \{1,2,3\},\ a \in \{1,2,3\}^{d-2}. \tag{4.3}$$

Indeed, the corresponding band totals satisfy

$$\sum_{i,j:\max(i,j)=k} b_{ij} = \ell_k = \sum_a d_{ka} \qquad (k = 1, 2, 3).$$

A zero denominator therefore corresponds to zero mass on both sides and is omitted. Summing (4.3) over a recovers b_{ij}, and summing over i, j recovers $\sum_k d_{ka}$. Thus the marginal masses are the prescribed coordinate interval lengths. Spreading these masses uniformly over their cells gives the interpolant in (2.5).

The following example illustrates the projection and gluing construction for two noncomparable points in dimension three.

Example 4.3. Take

$$\mathbf{x}^{(1)} = (0.3, 0.4, 0.8), \qquad \mathbf{x}^{(2)} = (0.6, 0.7, 0.5), \qquad q_1 = 0.15, \quad q_2 = 0.25.$$

Both lower Fréchet bounds are zero, and the upper bounds are 0.3 and 0.5. The two directed bounds hold because

$$q_1 - q_2 = -0.1 \le \sum_{i=1}^{3} (x_i^{(1)} - x_i^{(2)})_+ = 0.3, \qquad q_2 - q_1 = 0.1 \le \sum_{i=1}^{3} (x_i^{(2)} - x_i^{(1)})_+ = 0.6.$$

The first two coordinates have the same increasing order, while the third coordinate decreases. Formula (3.1) gives

$$z_1 = \max\{0.15,\ 0.25 - (0.6 - 0.3) - (0.7 - 0.4),\ 0.3 + 0.4 - 1\} = 0.15,$$
$$z_2 = \max\{0.15,\ 0.25,\ 0.6 + 0.7 - 1\} = 0.3.$$

Thus B must interpolate the values $0.15, 0.3$ at $(0.3, 0.4), (0.6, 0.7)$, while D must interpolate $0.15, 0.25$ at $(0.15, 0.8), (0.3, 0.5)$. The reduced points remain noncomparable.

For B, choose the nonnegative mass matrix

$$(b_{ij}) = \frac{1}{20} \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 5 & 3 & 0 \end{pmatrix}$$

on the coordinate grids $(0, 0.3, 0.6, 1)$ and $(0, 0.4, 0.7, 1)$. Each entry is the probability mass of a cell:

$$b_{ij} = \mu_B(J_{1i} \times J_{2j}) \qquad (i, j = 1, 2, 3).$$

The rows $i = 1, 2, 3$, displayed from top to bottom, correspond in increasing order to the first-coordinate intervals $[0, 0.3)$, $[0.3, 0.6)$, and $[0.6, 1)$. The columns $j = 1, 2, 3$, displayed from left to right, correspond to the second-coordinate intervals $[0, 0.4)$, $[0.4, 0.7)$, and $[0.7, 1)$. Thus the cell adjacent to the geometric origin, $[0, 0.3) \times [0, 0.4)$, has mass $b_{11} = 3/20$, which appears in the top-left entry of the matrix. The bottom-left entry $b_{31} = 5/20$ is the mass of $[0.6, 1) \times [0, 0.4)$. The same row and column convention is used for the other matrices below.

Its row sums are $(0.3, 0.3, 0.4)$ and its column sums are $(0.4, 0.3, 0.3)$. Also,

$$B(0.3, 0.4) = b_{11} = 0.15, \qquad B(0.6, 0.7) = \sum_{i,j \le 2} b_{ij} = 0.3.$$

For D, choose

$$(d_{ka}) = \frac{1}{20} \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 5 & 4 \end{pmatrix}$$

on the grids $(0, 0.15, 0.3, 1)$ and $(0, 0.5, 0.8, 1)$. Its row sums are $(0.15, 0.15, 0.7)$ and its column sums are $(0.5, 0.3, 0.2)$. Its specified values are

$$D(0.15, 0.8) = d_{11} + d_{12} = 0.15, \qquad D(0.3, 0.5) = d_{11} + d_{21} = 0.25.$$

Both matrices therefore define suitable bivariate checkerboard copulas by Lemma 2.3. The gluing bands have probabilities

$$\ell_1 = 0.15, \qquad \ell_2 = 0.15, \qquad \ell_3 = 0.7.$$

The final coordinate intervals are

i	J_{i1}	J_{i2}	J_{i3}
1	$[0, 0.3)$	$[0.3, 0.6)$	$[0.6, 1)$
2	$[0, 0.4)$	$[0.4, 0.7)$	$[0.7, 1)$
3	$[0, 0.5)$	$[0.5, 0.8)$	$[0.8, 1)$

Write $\mathsf{M}_k = (m_{ijk})_{i,j=1}^3$, $k = 1, 2, 3$, for the matrices of masses in the three third-coordinate intervals. Applying (4.3) gives

$$\mathsf{M}_1 = \frac{1}{280}\begin{pmatrix} 42 & 0 & 15 \\ 0 & 28 & 15 \\ 25 & 15 & 0 \end{pmatrix}, \quad \mathsf{M}_2 = \frac{1}{280}\begin{pmatrix} 0 & 0 & 15 \\ 0 & 14 & 15 \\ 25 & 15 & 0 \end{pmatrix}, \quad \mathsf{M}_3 = \frac{1}{280}\begin{pmatrix} 0 & 0 & 12 \\ 0 & 0 & 12 \\ 20 & 12 & 0 \end{pmatrix}.$$

For instance, the bivariate cell $(1, 3)$ belongs to band 3, so

$$m_{131} = \frac{b_{13} d_{31}}{\ell_3} = \frac{(3/20)(5/20)}{7/10} = \frac{3}{56} = \frac{15}{280}.$$

This mass is spread uniformly over $[0, 0.3) \times [0.7, 1) \times [0, 0.5)$. The three marginal mass vectors are $(0.3, 0.3, 0.4)$, $(0.4, 0.3, 0.3)$, and $(0.5, 0.3, 0.2)$, equal to the respective interval lengths. The resulting checkerboard copula has the required values:

$$C(\mathbf{x}^{(1)}) = m_{111} + m_{112} = \frac{42}{280} = 0.15, \qquad C(\mathbf{x}^{(2)}) = \sum_{i,j \le 2} m_{ij1} = \frac{42 + 28}{280} = 0.25.$$

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

In preparing this work, the authors used OpenAI's GPT-6 Pro to organize LaTeX drafts and check elementary estimates. The authors subsequently reviewed and verified all mathematical arguments and take full responsibility for the content of the manuscript.

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